# Math 22 –
# Linear Algebra and its applications

# - Lecture 25 -

**Instructor:** Bjoern Muetzel

# GENERAL INFORMATION

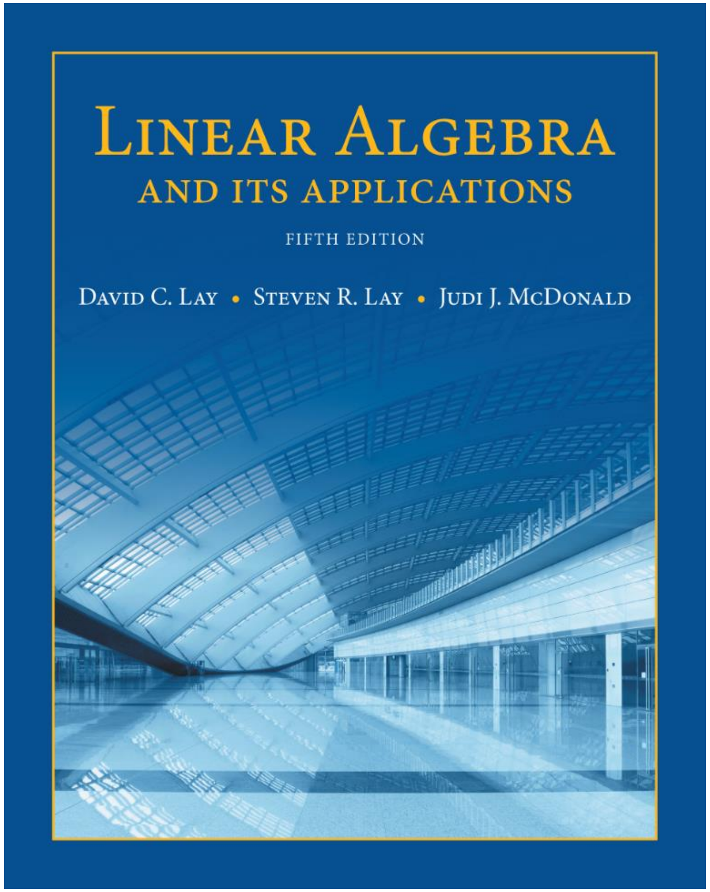
- **Office hours:** Tu 1-3 pm, **Th**, Sun **2-4 pm** in **KH 229**

  **Tutorial:** **Tu**, **Th**, Sun **7-9 pm** in **KH 105**

- **Homework 8**: due **Wednesday** at **4 pm** outside **KH 008.** There is only **Section B,C** and **D.**

# 5 Eigenvalues and Eigenvectors

## 5.1 EIGENVECTORS AND EIGENVALUES



**Summary:**

Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, then there is always a **good basis** on which the **transformation** has a **very simple form**.

To find this basis we have to find the **eigenvalues of *T*.**

# GEOMETRIC INTERPRETATION

**Example:** Let $A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}$ and let $u = x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

1.) Find $Av$ and Au.

Draw a picture of $v$ and $Av$ and $u$ and A$u$.

2.) Find $A(3u + 2v)$ and $A^2 (3u + 2v)$. **Hint:** Use part 1.)

# EIGENVECTORS AND EIGENVALUES

- **Definition:** An **eigenvector** of an $n \times n$ matrix $A$ is a **nonzero** vector **x** such that

  $$\boxed{Ax = \lambda x}$$ for some scalar λ in ℝ.

  In this case **λ** is called an **eigenvalue** and the solution **x≠ 0** is called an **eigenvector corresponding to λ**.

- **Definition:** Let $A$ be an $n \times n$ matrix. The set of solutions

  $$\boxed{\mathbf{Eig}(A, \lambda) = \{x \text{ in } \mathbb{R}^n, \text{ such that } (A - \lambda I_n)x = 0\}}$$

  is called the **eigenspace Eig(*A*, λ)** of $A$ corresponding to λ.

  It is the null space of the matrix $A - \lambda I_n$:

  $$\boxed{\mathrm{Eig}(A, \lambda) = \mathrm{Nul}(A - \lambda I_n)}$$

# EIGENVECTORS AND EIGENVALUES

**Example:** Show that $\lambda = 7$ is an eigenvalue of matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

and find the corresponding eigenspace $\text{Eig}(A, 7)$.

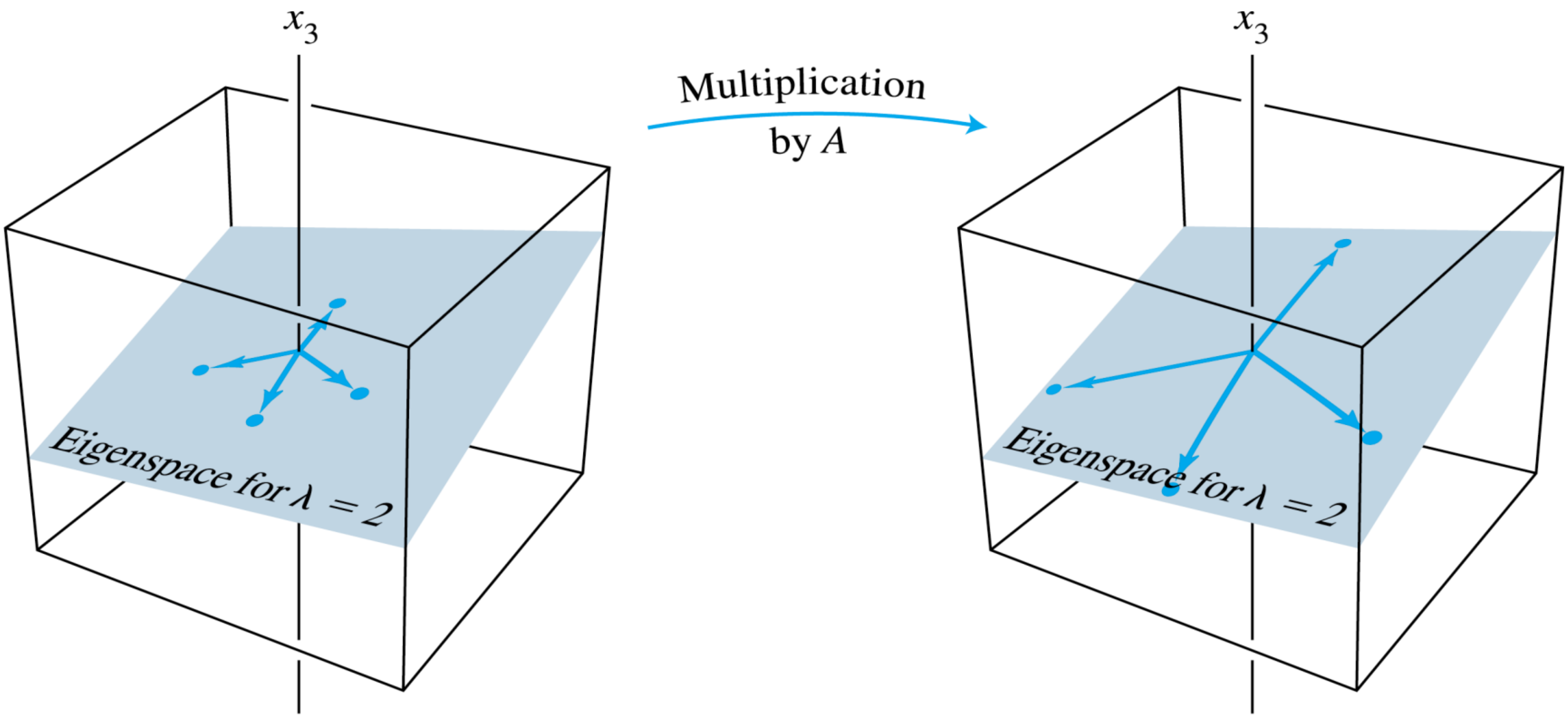
# EIGENVECTORS AND EIGENVALUES

- **Example:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of $A$ is $\lambda = 2$.

  Find a basis for the corresponding eigenspace $\text{Eig}(A,2)$.

# EIGENVECTORS AND EIGENVALUES

- The eigenspace $\operatorname{Eig}(A,2) = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}$ is a subspace of $\mathbb{R}^3$.




$A$ acts as a dilation on the eigenspace.

# THEOREMS ABOUT EIGENVALUES

- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

- **Warning:** We can **not** find the eigenvalues of a matrix $A$ by row reducing to echelon form $U$. As ***A* and *U* have** usually **different eigenvalues.**

- **Proof of Theorem 1:**

# THEOREMS ABOUT EIGENVALUES

# THEOREMS ABOUT EIGENVALUES

- **Theorem 2:** If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix $A$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

- **Proof:** Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent.

- Then there is a subset of $S=\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ , say $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ that is a **basis** for Span(S) and a vector, say $\mathbf{v}_{p+1}$ that is a linear combination of these vectors.

- Then there exist scalars $c_1, \ldots, c_p$ such that

$$c_1 v_1 + \cdots + c_p v_p = v_{p+1} \quad (1)$$

- Multiplying both sides of (1) by $A$ and using the fact that $Av_k = \lambda_k v_k$ for each $k$, we obtain by the linearity of $A$

or

(2)

- Multiplying both sides of (1) by $\lambda_{p+1}$ and substituting the result in the right hand side of (2), we obtain

or

$= 0.$ (3)

- Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent, the weights in (3) are all zero. But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence $c_i = 0$ for $i = 1, \ldots, p$.

- But then (1) states that $v_{p+1} = 0$, which is impossible.

# EIGENVECTORS AND DIFFERENCE EQUATIONS

**Application to a recursive sequence in $\mathbb{R}^n$**

Let $A$ be an $n \times n$ matrix and consider the **recursive sequence** $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ given by $x_0 = u$ in $\mathbb{R}^n$ and

$$\boxed{x_{k+1} = Ax_k} \qquad \text{for } k = 0,1,2,3,\dots,$$

**Definition:** We call a **solution** of this equation an explicit description of $\{x_k\}$ whose formula for each $x_k$ does **not depend directly on $A$** or on the preceding terms in the sequence other than the initial term $\boldsymbol{x_0 = u}$.

**Note:** It follows that

$$x_{k+1} = A^k x_0 = A^k u,$$

However, this is **not explicit enough** to be a solution.

## Proof of the Note:

# EIGENVECTORS AND DIFFERENCE EQUATIONS

- **Example:** Let A be an $n \times n$ matrix such that

$$A b_1 = 2 b_1 \qquad \text{and} \qquad A b_2 = \frac{1}{3} b_2 \qquad \text{where} \qquad b_1, b_2 \neq 0.$$

1.) Calculate $A^2 b_1$ and $A^2 b_2$.

2.) Calculate $A^k b_1$ and $A^k b_2$ and describe geometrically what happens to $A^k b_1$ and $A^k b_2$.

3.) Find a formula for $A^k(4b_1+5b_2)$.