Math 22 –
Linear Algebra and its applications

- Lecture 26 -

Instructor: Bjoern Muetzel

GENERAL INFORMATION

- **Office hours:** Tu 1-3 pm, Th, **Sun 2-4 pm** in **KH 229**

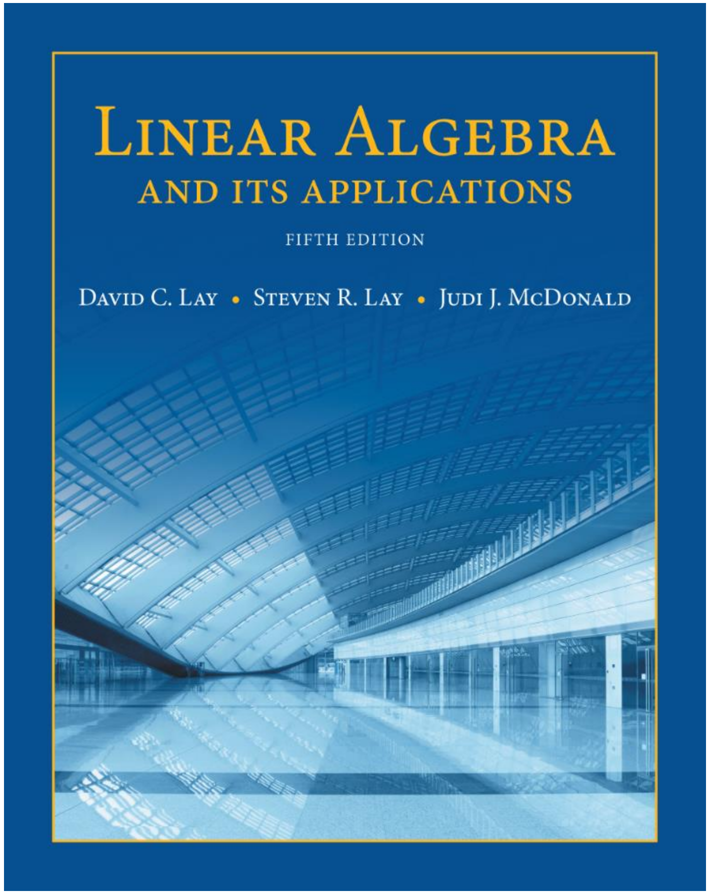
  Tutorial: Tu, Th, **Sun 7-9 pm** in **KH 105**

- **Homework 8**: due **Wednesday** at **4 pm** outside **KH 008.** Please give in **part B, C** and **D.** There is **no part A.**

5 Eigenvalues and Eigenvectors

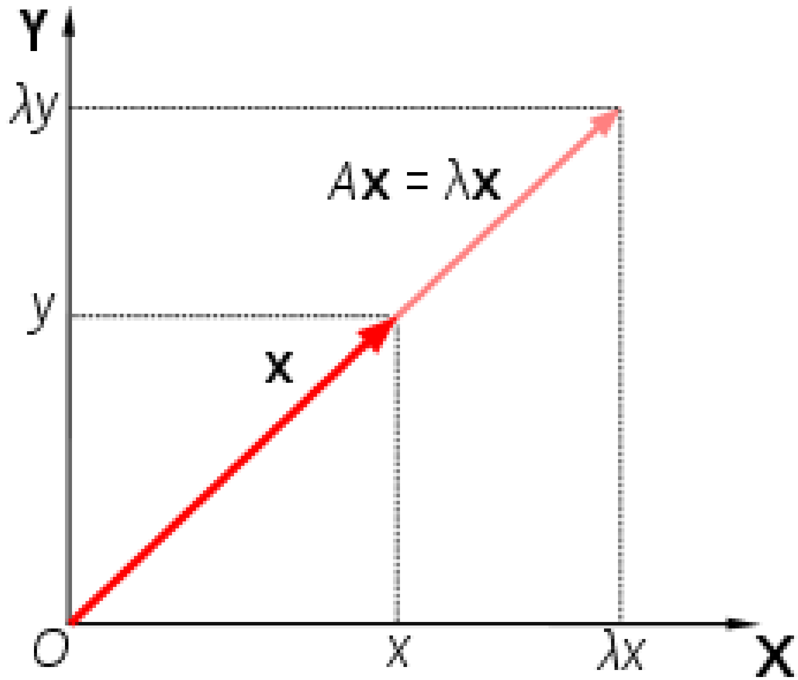
5.2

THE CHARACTERISTIC EQUATION



Summary:

We can use **determinants** to find the **eigenvalues** of a matrix *A*. Finding the **eigenvalues** of a matrix amounts to **finding the roots** of the **characteristic polynomial**.



REVIEW: DETERMINANTS

Theorem 3: (Properties of Determinants)

If A and B are $n \times n$ matrices, then

a. A is **invertible** if and only if $\det(A) \neq 0$.

b. $\det(AB) =$.

c. $\det A^T =$.

d. $\det(A^{-1}) =$.

Proof of Theorem 3d:

REVIEW: DETERMINANTS

- **Defintion:** Given $A = [a_{ij}]$, the **(*i, j*)-cofactor** of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \, detA_{ij} \quad .$$

The **sign of the cofactor** can be read from the **sign matrix**:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \ddots \end{bmatrix} \qquad \boldsymbol{A_{21}} = \begin{bmatrix} & a_{1,2} & \cdots & a_{1,n} \\ & & & \\ & \vdots & \ddots & \vdots \\ & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}.$$

Example: $C_{21} = (-1)^{2+1} detA_{21}$

REVIEW: DETERMINANTS

Theorem: **(Cofactor expansion)**

The determinant of an $n \times n$ matrix A can be computed by a cofactor across **any row** or down **any column**.

The **expansion** across the ***i*-th row** using cofactors is

$$detA = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The **expansion** down the ***j*-th column** is

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example: Use a cofactor expansion down the third column to compute det *A*, where

$$A = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 2 & 4 & 2 & 6 \\ 4 & -2 & 0 & -3 \\ 0 & 7 & 0 & -2 \end{bmatrix}$$

THE CHARACTERISTIC EQUATION

How can we find the eigenvalues of a matrix *A* ?

- Let A be an $n \times n$ matrix. We know that the eigenvectors for a certain eigenvalue λ lie in the null space

$$\boxed{\mathbf{Nul}(\boldsymbol{A}-\boldsymbol{\lambda I_n}) = \{\mathrm{x} \text{ in } \mathbb{R}^n, \text{ such that } (A-\lambda I_n)\mathrm{x} = 0\} = \mathbf{Eig}(\boldsymbol{A}, \boldsymbol{\lambda})}$$

- This means that $A - \lambda I_n$ is not invertible or
$$\boxed{\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda I_n}) = \mathbf{0}}.$$

- Hence we can **find** the **eigenvalues of** ***A*** by **solving the equation** $\boxed{\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda I_n}) = \mathbf{0}}$ **for** $\boldsymbol{\lambda}$**.**

THE CHARACTERISTIC EQUATION

- **Example:** Find the characteristic equation of *A* and determine the eigenvalues of *A*. Then find the eigenspaces associated to these eigenvalues.

$$A = \begin{bmatrix} 5 & -8 & 0 \\ 0 & 5 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

THE CHARACTERISTIC EQUATION

Theorem: Let A be $n \times n$ matrix. The equation

$$\boxed{\det(A - \lambda I_n) = 0}$$

is called the **characteristic equation of *A***.

Furthermore **λ in** $\mathbb{R}$ is an **eigenvalue of *A*** if and only if λ satisfies the characteristic equation.

Definition: If A is an $n \times n$ matrix, then

1.) $\det(A - \lambda I_n)$ is a polynomial of degree n called the **characteristic polynomial** of A.

2.) The (**algebraic**) **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Example:

SIMILARITY

- **Definition:** If A and B are $n \times n$ matrices, then A **is similar to** B if there is an invertible matrix P such that

$$\boxed{A = PBP^{-1}} \quad \text{or, equivalently} \quad \boxed{P^{-1}AP = B}.$$

- Setting Q= P^{-1}, we have $B = Q^{-1}AQ$.

  So B is also similar to A, and we say simply that A and B **are similar**.

- Changing A into PAP^{-1} is called a **similarity transformation**.

- **Theorem 4:** If $n \times n$ matrices A and B are **similar**, then they have the **same characteristic polynomial** and hence the **same eigenvalues** with the **same multiplicities**.

- **Proof:**

Warning:

1.) The matrices $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are **not similar** even though they **have** the **same eigenvalues.**

2.) Similarity is not the same as row equivalence. **Row operations** on a matrix usually **change** its **eigenvalues.**

SIMILARITY

Example: For $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 6 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 1/6 \\ 0 & 1/2 & -1/6 \\ 0 & 0 & 1/3 \end{bmatrix} = PBQ$.

1.) Use **Theorem 3** to calculate $\det A$.

2.) Is $Q = P^{-1}$?

3.) Find the eigenvalues of A using **Theorem 4**

4.) Can you find an eigenvector of A without solving the equation for the eigenspace ? **Hint:** Look at BQ