Math 22 - Fall 2019
Midterm Exam 1 - Solutions

Your name: ______________________________

Section (please check the box): ☐ Section 1 (10 hour) ☐ Section 2 (2 hour)

INSTRUCTIONS

- Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer.
- It is fine to leave your answer in a form such as $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $\cos(\pi)$ or $(3-2)$), you should simplify it.
- You may use the last two pages for scrap paper.
- This is a closed book exam. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource.

GOOD LUCK!

(1) (*12 points*) Please indicate whether the following statements are **TRUE** or **FALSE**. **Circle** the correct answer.

1.) If the coefficient matrix A of an augmented matrix $[A|\mathbf{b}]$ has a nonpivot column then the corresponding system of linear equations has infinitely many solutions.

TRUE $\boxed{\textbf{FALSE}}$

Solution: This is false. Even if A has a nonpivot column, there could still be a last row $[0\ 0 \ldots 0|b_n]$ in the echelon form which makes the system inconsistent.

2.) The matrix $\begin{bmatrix} 0 & 1 & -3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form (ref).

TRUE $\boxed{\textbf{FALSE}}$

Solution: This is false. If the matrix was in reduced echelon form, there would be a 0 above the pivot in column 4.

3.) Let $[A|\mathbf{b}]$ be an augmented matrix. If A has a pivot in each row then the corresponding system of linear equations has a solution.

$\boxed{\textbf{TRUE}}$ **FALSE**

Solution: This is true. This follows from a theorem from the lecture.

4.) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors. Then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

$\boxed{\textbf{TRUE}}$ **FALSE**

Solution: This is true. This is the definition of a span.

5.) If T is a linear transformation, and A is the standard matrix for T, then T is one-to-one if and only if A has a pivot in every column.

$\boxed{\textbf{TRUE}}$ **FALSE**

Solution: This is true. Again this follows from a theorem from the lecture.

6.) The transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is linear.

TRUE $\boxed{\textbf{FALSE}}$

Solution: This is false. If T is a linear transformation then we must have $T(\mathbf{0}) = \mathbf{0}$. However, in this case we have $T(\mathbf{0}) = [1, 2]$. So T can not be linear.

(2) (*16 points*) Let

$$A = \begin{bmatrix} 1 & -3 & 4 & -3 & 2 \\ 3 & -7 & 8 & -5 & 8 \\ 0 & 3 & -6 & 6 & 4 \end{bmatrix} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5],$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ and $\mathbf{a}_5$ denote the columns of A.

a) Solve the matrix equation $A\mathbf{x} = \mathbf{0}$, and write the solution in parametric vector form.

Solution: To solve this equation we have to row reduce the matrix $[A|\mathbf{0}]$:

$$[A|\mathbf{0}] = \left[\begin{array}{ccccc:c} 1 & -3 & 4 & -3 & 2 & 0 \\ 3 & -7 & 8 & -5 & 8 & 0 \\ 0 & 3 & -6 & 6 & 4 & 0 \end{array}\right] \overset{-3R1+R2}{\rightarrow} \left[\begin{array}{ccccc:c} 1 & -3 & 4 & -3 & 2 & 0 \\ 0 & 2 & -4 & 4 & 2 & 0 \\ 0 & 3 & -6 & 6 & 4 & 0 \end{array}\right] \overset{\frac{1}{2}R2}{\rightarrow}$$

$$\left[\begin{array}{ccccc:c} 1 & -3 & 4 & -3 & 2 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 6 & 4 & 0 \end{array}\right] \overset{-3R2+R3}{\rightarrow} \left[\begin{array}{ccccc:c} 1 & -3 & 4 & -3 & 2 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \overset{-R3+R2,-2R3+R1}{\rightarrow}$$

$$\left[\begin{array}{ccccc:c} 1 & -3 & 4 & -3 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 \\ 0 & 3 & -6 & 6 & 4 & 0 \end{array}\right] \overset{3R2+R1}{\rightarrow} \left[\begin{array}{ccccc:c} \boxed{1} & 0 & -2 & 3 & 0 & 0 \\ 0 & \boxed{1} & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right].$$

Reading out the equations we obtain

$$\begin{array}{rccc} x_1 = & 2x_3 & -3x_4 \\ x_2 = & 2x_3 & -2x_4 \\ x_3 = & x_3 & \\ x_4 = & & x_4 \\ x_5 = & 0 & \end{array} \quad \text{or} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{,where } s, t \in \mathbb{R}.$$

b) We know that the vector $\frac{1}{2}\mathbf{e}_3$ is a solution to the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \boxed{\mathbf{b}}$. Write down all the solutions to the equation $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

Solution: We know from the lecture that if $\frac{1}{2}\mathbf{e}_3$ is a solution for $A\mathbf{x} = \mathbf{b}$ then any solution for this equation is $\frac{1}{2}\mathbf{e}_3 + \mathbf{v}_h$, where $\mathbf{v}_h$ is a solution for $A\mathbf{x} = \mathbf{0}$. So with part **a)** we have:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{,where } s, t \in \mathbb{R}.$$

c) Is $\mathbf{a}_5$ in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$? Justify your answer.

Solution: To solve this equation we have to row reduce the matrix $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4|\mathbf{a}_5]$:

$$[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4|\mathbf{a}_5] = \left[\begin{array}{cccc|c} 1 & -3 & 4 & -3 & 2 \\ 3 & -7 & 8 & -5 & 8 \\ 0 & 3 & -6 & 6 & 4 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 0 & -2 & 3 & 0 \\ 0 & \boxed{1} & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right].$$

Luckily we already did the row reduction in part **a)** and can use this result. Looking at the equations we see that there is a contradiction in the third equation ($0 = 1$). So $\mathbf{a}_5$ is not in the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

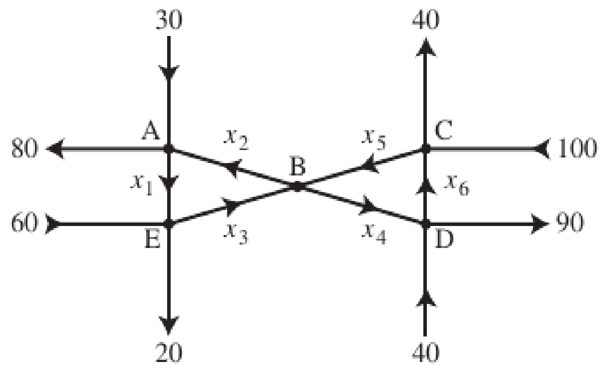
d) Is the range of the linear map T, such that $T(\mathbf{x}) = A\mathbf{x}$ all of $\mathbb{R}^3$? Justify your answer.

Solution: This is equivalent to the statement that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ in $\mathbb{R}^3$. This again is equal to that A has a pivot in every row.

$$A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5] = \begin{bmatrix} 1 & -3 & 4 & -3 & 2 \\ 3 & -7 & 8 & -5 & 8 \\ 0 & 3 & -6 & 6 & 4 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & -2 & 3 & 0 \\ 0 & \boxed{1} & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

From part **a)** we see that indeed A has a pivot in every row. So the range of T is all of $\mathbb{R}^3$.

(3) (*12 points*) We want to analyze the traffic flow of the freeway network below where the numbers are given by cars per minute.



a) Write down the equations that describe the flow of cars in this system.

Solution: Noting that the flow in must equal the flow out at each intersection, we write down the equations for each intersection. The same must be true for the total in and outflow.

Intersection	Flow in	Flow out
A	$x_2 + 30$	$x_1 + 80$
B	$x_3 + x_5$	$x_2 + x_4$
C	$x_6 + 100$	$x_5 + 40$
D	$x_4 + 40$	$x_6 + 90$
E	$x_1 + 60$	$x_3 + 20$
Total	230	230

At each intersection the left-hand side must be equal to the right-hand side.

b) Write down the augmented matrix of the system of linear equations from part **a)**.
You do **not** have to solve the equations.

Solution: After rearranging the equations for the intersections by variables we obtain a system of linear equations in the variables x_1, x_2, x_3, x_4, x_5 and x_6. This system has the augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{array}\right].$$

(4) (*12 points*) Consider the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ c^2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ c \end{bmatrix}$, where c is a constant.

For example, if $c = 2$, then $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

a) For which values of c is $\mathbf{b}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$?

Solution: To find out if $\mathbf{b}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ we have to check if there are numbers x_1, x_2, such that $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{b}$. We can check if this system of linear equations has a solution by looking at the echelon form of the augmented matrix $[\mathbf{u}, \mathbf{v}|\mathbf{b}]$. Row reducing this matrix we obtain

$$[\mathbf{u}, \mathbf{v}|\mathbf{b}] = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & c^2 & c \end{array}\right] \overset{-R1+R2}{\to} \left[\begin{array}{cc:c} \boxed{1} & 1 & 1 \\ 0 & c^2-1 & c-1 \end{array}\right].$$

We see that if $c^2 \neq 1$ then we have a pivot in every row and the equation must have a solution. It remains to look at the cases $c = 1$ and $c = -1$.

$$\textbf{Case 1} \quad c = 1 : [\mathbf{u}, \mathbf{v}|\mathbf{b}] = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc:c} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\textbf{Case 2} \quad c = -1 : [\mathbf{u}, \mathbf{v}|\mathbf{b}] = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cc:c} \boxed{1} & 1 & 1 \\ 0 & 0 & -2 \end{array}\right].$$

So in the second case there is no solution. In total we have that $\mathbf{b}$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$ if $\boxed{c \neq -1}$.

b) For which values of c is $\text{Span}\{\mathbf{u}, \mathbf{v}\} = \mathbb{R}^2$?

Solution: We have to check for which values of c the matrix $[\mathbf{u}, \mathbf{v}]$ has a pivot in every row. From **a)** we have

$$[\mathbf{u}, \mathbf{v}] = \begin{bmatrix} 1 & 1 \\ 1 & c^2 \end{bmatrix} \overset{-R1+R2}{\to} \begin{bmatrix} \boxed{1} & 1 \\ 0 & c^2-1 \end{bmatrix}.$$

So $[\mathbf{u}, \mathbf{v}]$ has a pivot in every row if $c^2 \neq 1$ or $\boxed{c \neq 1}$ and $\boxed{c \neq -1}$.

c) Are there any values of c for which the vector equation $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{b}$ has infinitely many solutions?

Solution: Again from **a)** we see that this is true if $\boxed{c = 1}$. This is the only case where the system of linear equations has a free variable.

(5) (*14 points*) Consider the following closed economy with the three S_1, S_2 and S_3.

Sector one consumes 90 percent of it's own product, sells 10 percent to sector two and nothing to sector three. Sector two consumes 40 percent of it's own product, sells 40 percent to sector one and 20 percent to sector three. Sector three consumes 20 percent of it's own product, sells 40 percent to sector 1 and 40 percent to sector 2.

a) Write a set of equations that describes, for each sector, the total cost of products bought, c_1, c_2, and c_3, in terms of the variables P_1, P_2, and P_3 (which represent the total price of goods produced by sectors 1, 2, and 3 respectively).

Solution:

$$\begin{aligned} c_1 &= 0.9P_1 && +0.4P_2 && +0.4P_3 \\ c_2 &= 0.1P_1 && +0.4P_2 && +0.4P_3 \\ c_3 &= && +0.2P_2 && +0.2P_3 \end{aligned} .$$

b) If the economy is in equilibrium, write a system of linear equations that describes the relationships between the prices P_1, P_2, and P_3.

Solution: Replacing c_i for P_i in the previous system of equations produces a new system of equations involving the equilibrium prices:

$$\begin{aligned} 0 &= -0.1P_1 && +0.4P_2 && +0.4P_3 \\ 0 &= 0.1P_1 && -0.6P_2 && +0.4P_3 \\ 0 &= && +0.2P_2 && -0.8P_3 \end{aligned} .$$

c) Describe (e.g., parametrize) the possible equilibrium prices.

Solution: The system of equations in problem 2. can be solved by the standard technique:

$$\left[\begin{array}{ccc|c} -0.1 & 0.4 & 0.4 & 0 \\ 0.1 & -0.6 & 0.4 & 0 \\ 0 & 0.2 & -0.8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 4 & 4 & 0 \\ 1 & -6 & 4 & 0 \\ 0 & 2 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -20 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, the possible equilibrium prices can be expressed in terms of the free variable P_3.

$$\boxed{\begin{aligned} P_1 &= 20P_3 \\ P_2 &= 4P_3 \\ P_3 &= P_3 \end{aligned}} .$$

(6) (*12 points*) Let T be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 5x_1 + 2x_2 + x_3 \\ x_2 - 3x_3 \\ x_3 \\ x_3 \end{bmatrix}.$$

a) What is the domain and the codomain of T?

Solution: The domain or source space of T is $\mathbb{R}^3$.
The codomain or target space of T is $\mathbb{R}^4$.

b) Find the standard matrix for T.

Solution: There are different ways of finding the solution. One way is to look at the images of the unit vectors. We have

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}.$$

By a theorem from the lecture the standard matrix of T is

$$A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)] = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

c) Is T onto? Justify your answer.

Solution: By the lecture T is onto if A has a pivot in every row. We check

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{5} & 2 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

So A does not have a pivot in the last row and therefore T is not onto.

d) Is T one-to-one? Justify your answer.

Solution: By the lecture T is one-to-one if A has a pivot in every column. From part **c)** we see that this is true. So T is one-to-one.

(7) (*12 points*) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that

1.) first rotates points (about the origin) counterclockwise by an angle of $\frac{3}{2}\pi$ radians,
2.) then performs a vertical shear that transforms $\mathbf{e}_1$ into $\mathbf{e}_1 + 4\mathbf{e}_2$ (leaving $\mathbf{e}_2$ unchanged),
3.) finally reflects points through the x_1 axis.

Explain why T is a linear map. Then find the standard matrix of T. Explain all your steps.
Hint: For this question it might be useful to draw the situation on a scrap paper or below.

Solution: T is a linear map as it is a composition of three linear maps. Any composition of linear maps is again a linear map.

To find the standard matrix of T we find the images of the unit vectors for the three maps of which T is composed of. Let $R_{\frac{3}{2}\pi}$ be the matrix of the rotation, V be the matrix of the vertical shear and M be the matrix of the reflection.

i) It is easy to see by drawing a picture that

$$R_{\frac{3}{2}\pi} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; R_{\frac{3}{2}\pi} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ therefore } R_{\frac{3}{2}\pi} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

ii) For the vertical shear V we find by 2.)

$$V \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \; V \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ therefore } V = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}.$$

iii) Finally for the mirror reflection M we have

$$M \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; M \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \text{ therefore } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In total we get, that $T = M \circ V \circ R_{\frac{3}{2}\pi}$. For its standard matrix A this means

$$A = M \cdot V \cdot R_{\frac{3}{2}\pi} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}}.$$

(8) (*10 points*) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly dependent vectors in $\mathbb{R}^n$. Prove that the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)$ are also linearly dependent.

Solution: We know that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly dependent, so there are numbers $c_1, c_2, \ldots, c_p$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = \mathbf{0} \quad (1).$$

We want to show that the same is true for the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)$.

To this end we apply the map T to both sides of equation (1). We obtain:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p) = T(\mathbf{0}) = \mathbf{0},$$

as the image of the zero vector in $\mathbb{R}^n$ is the zero vector in $\mathbb{R}^m$. By linearity it follows from this equation that

$$\begin{aligned} T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p) &= \mathbf{0} \text{ hence} \\ c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_pT(\mathbf{v}_p) &= \mathbf{0}. \end{aligned}$$

We already know that the weights $c_1, c_2, \ldots, c_p$ are not all zero. So the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)$ are also linearly independent.