Math 22 - Fall 2019
Midterm Exam 2

Your name: ______________________________

Section (please check the box): ☐ Section 1 (10 hour) ☐ Section 2 (2 hour)

INSTRUCTIONS

- Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer.
- It is fine to leave your answer in a form such as $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $\cos(\pi)$ or $(3-2)$), you should simplify it.
- You may use the last page for scrap paper.
- This is a closed book exam. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource.

GOOD LUCK!

(1) *(12 points)* Please indicate whether the following statements are **TRUE** or **FALSE**.
Circle the correct answer. You do not have to show your work, however thinking about the problem on a scrap paper is recommended.

1.) If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $B = \begin{bmatrix} a & -c & b \\ d & -f & e \\ g & -i & h \end{bmatrix}$ then $\det(A) = \det(B)$.

$\boxed{\textbf{TRUE}}$ **FALSE**

Solution: This is true. The matrix B results from the matrix A by swapping one column and then multiplying column 2 by -1. As $\det(A) = \det(A^T)$ and a row swap changes the determinant by a factor of -1, so does a column swap. The same is true for multiplying a column by -1. So in total $\det(A) = (-1)(-1)\det(B) = \det(B)$.

2.) The set $W = \{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$, such that $x_1 \geq 0\}$ is a subspace of $\mathbb{R}^2$.

TRUE $\boxed{\textbf{FALSE}}$

Solution: This is false, as $\mathbf{e}_1 \in W$, but $-\mathbf{e}_1 = (-1)\mathbf{e}_1$ not in W.

3.) For every 2×4 matrix A we have $\dim(\mathrm{Nul}(A)) = \mathrm{Rank}(A)$.

TRUE $\boxed{\textbf{FALSE}}$

Solution: This is false, as for $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we have $\dim(\mathrm{Nul}(A)) = 4$ and $\mathrm{Rank}(A) = 0$.

4.) There exists a 2×4 matrix A such that $\dim(\mathrm{Nul}(A)) = \mathrm{Rank}(A)$.

$\boxed{\textbf{TRUE}}$ **FALSE**

Solution: This is true. For $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, we have $\dim(\mathrm{Nul}(A)) = \mathrm{Rank}(A)$.

5.) Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of a vector space V. Then $[\mathbf{b}_1]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[\mathbf{b}_2]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\boxed{\textbf{TRUE}}$ **FALSE**

Solution: This is true, as $\mathbf{b}_1 = 1 \cdot \mathbf{b}_1 + 0 \cdot \mathbf{b}_2$ and $\mathbf{b}_2 = 0 \cdot \mathbf{b}_1 + 1 \cdot \mathbf{b}_2$.

6.) Let H be a subset of $\mathbb{P}_2$, the polynomials of degree at most 2, defined by

$$H = \{\mathbf{p}(t) \text{ in } \mathbb{P}_2 : \mathbf{p}(1) = 0\}.$$

Then H is a subspace of $\mathbb{P}_2$.

$\boxed{\textbf{TRUE}}$ **FALSE**

Solution: This is true. We check the three subspace criteria.

(2) *(12 points)* Let A and B be the two matrices given by

$$A = \begin{bmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 2 & -1 & 5 \\ 0 & 4 & 4 & 10 \\ 0 & 0 & 3 & 5 \end{bmatrix}.$$

Find the determinant of the matrices described below. Use the method that is appropriate for the problem.

a) Find $\det(A)$.

Solution: We can use the cofactor expansion of $\det(A)$ along the third row. Then

$$\det(A) = 3 \cdot \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} \overset{\text{column } 3}{=} 3 \cdot 5 \cdot \begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix} = 3 \cdot 5 \cdot (7-6) = \boxed{15}.$$

b) Find $\det(B)$.

Solution: This matrix is almost in echelon form U. We can use the row reduction algorithm. To this end we row reduce B and record each step.

$$B = \begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 2 & -1 & 5 \\ 0 & 4 & 4 & 10 \\ 0 & 0 & 3 & 5 \end{bmatrix} \overset{-2R2+R3}{\longrightarrow} \begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \overset{\frac{1}{2}R3}{\longrightarrow} \begin{bmatrix} 1 & 2 & -3 & -5 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 5 \end{bmatrix} \overset{-R3+R4}{\longrightarrow} \begin{bmatrix} \boxed{1} & 2 & -3 & -5 \\ 0 & \boxed{2} & -1 & 5 \\ 0 & 0 & \boxed{3} & 0 \\ 0 & 0 & 0 & \boxed{5} \end{bmatrix}.$$

So $\det(U) = 1 \cdot 2 \cdot 3 \cdot 5 = 30$. The only row transformation that changed the determinant was the scaling in step 2. Thereofore we know that $\frac{1}{2}\det(A) = \det(U)$. So $\det(A) = 2 \cdot 30 = \boxed{60}$.

c) Find $\det(B - I_4)$, where I_4 is the 4×4 identity matrix.

Solution: We have that

$$B - I_4 = \begin{bmatrix} 0 & 2 & -3 & -5 \\ 0 & 1 & -1 & 5 \\ 0 & 4 & 3 & 10 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

The matrix $B - I_4$ has a zero column. Therefore it is not invertible. By a theorem from the lecture that means that $\det(B - I_4) = 0$.

d) For $C = A^{-1} \cdot B$, find $\det(C)$ and $\det(C^{100})$.

Solution: We know that

$$\det(C) = \det(A^{-1} \cdot B) = \det(A^{-1}) \cdot \det(B) = \frac{1}{\det(A)} \cdot \det(B) = \frac{60}{15} = \boxed{4}.$$

Then $\det(C^{100}) = \det(C)^{100} = \boxed{4^{100}}$.

(3) *(14 points)* Assume that A is row-equivalent to B.

$$A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5] = \begin{bmatrix} 0 & 2 & 0 & -2 & 6 \\ 1 & 2 & 3 & 0 & 0 \\ -1 & 0 & -3 & -1 & 5 \\ 2 & 7 & 6 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -2 & 6 \end{bmatrix}$$

a) Find a basis for $\mathrm{Col}(A)$.

Solution: We row reduce B (and therefore also A) to echelon form U to find the pivot columns of U and A:

$$B = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & -2 & 6 \end{bmatrix} \overset{-2R3+R4}{\longrightarrow} \begin{bmatrix} \boxed{1} & 2 & 3 & 0 & 0 \\ 0 & \boxed{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 5 \\ 0 & 0 & 0 & 0 & \boxed{-4} \end{bmatrix} = U\,.$$

Then a basis B_C for $\mathrm{Col}(A)$ are the pivot columns of A. So $B_C = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5\}$.

b) Find a basis for $\mathrm{Nul}(A)$.

Solution: To find a basis B_N of the null space of A we have solve the system of linear equations whose augmented matrix is $[A|\mathbf{0}]$. To this end we row reduce $[U|\mathbf{0}]$ (and also $[A|\mathbf{0}]$ to reduced echelon form.

$$[U|\mathbf{0}] = \left[\begin{array}{ccccc:c} \boxed{1} & 2 & 3 & 0 & 0 & 0 \\ 0 & \boxed{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} & 5 & 0 \\ 0 & 0 & 0 & 0 & \boxed{-4} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc:c} \boxed{1} & 0 & 3 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right] = [\mathrm{ref}A|\mathbf{0}].$$

From the reduced echelon form $[\mathrm{ref}A|\mathbf{0}]$ we obtain the set of solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \cdot \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{=\mathbf{v}_1}, \quad \text{where } x_3 \in \mathbb{R}. \text{ So } B_N = \{\mathbf{v}_1\}.$$

c) Find a basis for $\mathrm{Row}(A)$.

Solution: A basis B_R for the row space $\mathrm{Row}(A)$ of A are the row vectors of an echelon form of A. So, using part **c)** we obtain

$$B_R = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Alternative: Alternatively we could have also used the pivot row vectors of U in part **b)**.

(4) *(12 points)* Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the map given by

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}.$$

Example: If $p(t) = 1 + 2t$, then $T(p) = \begin{bmatrix} 1 + 2 \cdot 0 \\ 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

a) Show that T is a linear map.

Solution: We have to check the two conditions for a linear map:

1.) For every $p, q \in \mathbb{P}_2$ we have $T(p+q) = T(p) + T(q)$:

$$T(p+q) = \begin{bmatrix} p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = T(p) + T(q).$$

2.) For every $p \in \mathbb{P}_2$ and $c \in \mathbb{R}$, we have $T(c \cdot p) = c \cdot T(p)$:

$$T(c \cdot p) = \begin{bmatrix} c \cdot p(0) \\ c \cdot p(1) \end{bmatrix} = c \cdot \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = c \cdot T(p).$$

So T is a linear map.

b) Find a basis for the kernel or nullspace $\text{Nul}(T)$ of T.

Solution: We have to find

$$\text{Nul}(T) = \{p \in \mathbb{P}_2, \text{ such that } T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}.$$

We know that any polynomial $p \in Po_2$ can be written as $p(t) = a + bt + t^2$. So the condition for p being in $\text{Nul}(T)$ is

$$T(p) = \begin{bmatrix} a + b0 + c0^2 \\ a + b1 + c1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } a = 0 \text{ and } c = -b.$$

So $p \in \text{Nul}(T) \Leftrightarrow p(t) = b(t - t^2)$. So a basis for $\text{Nul}(T)$ is $\{t - t^2\}$.

c) What is the dimension of the range $T(\mathbb{P}_2)$ of T. Justify your answer.

Solution: We know that $\dim(\text{Nul}(T)) + \dim(T(\mathbb{P}_2)) = \dim(\mathbb{P}_2) = 3$.

So $\dim(T(\mathbb{P}_2) = 2$.

d) Find a basis for the range $T(\mathbb{P}_2)$ of T.

Solution: We know from part **c)** that $\dim(T(\mathbb{P}_2)) = 2$, so $T(\mathbb{P}_2) = \mathbb{R}^2$. A basis is

$E = \{\mathbf{e}_1, \mathbf{e}_2\}$.

Alternative: A polynomial in $\mathbb{P}_2$ can be written as $p(t) = a + bt + t^2$, so

$$T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = \begin{bmatrix} a \\ a + b + c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So $T(\mathbb{P}_2) = \text{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and a basis is $B = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$.

(5) *(12 points)* Let $W = \left\{ \begin{bmatrix} a+b+2c \\ 4a+2b+2c \\ -a-3b-8c \\ a+2b+5c \end{bmatrix}, \text{ where } a,b,c \text{ in } \mathbb{R} \right\}$ be a subset of $\mathbb{R}^4$.

a) Explain why W is a vector space. Justify your answer.

Solution: We have that

$$W = \left\{ a \cdot \underbrace{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}}_{=\mathbf{v}_1} + b \cdot \underbrace{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 2 \end{bmatrix}}_{=\mathbf{v}_2} + c \cdot \underbrace{\begin{bmatrix} 2 \\ 2 \\ -8 \\ 5 \end{bmatrix}}_{=\mathbf{v}_3}, \text{ where } a,b,c \text{ in } \mathbb{R} \right\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

Every span is a vector space, so W is a vector space.

b) Find a basis B_W for W. What is the dimension of W?

Solution: To this end we row reduce the matrix $A = [\mathbf{v}_1, \mathbf{v_2}, \mathbf{v}_3]$ to echelon form U. We find

$$A = [\mathbf{v}_1, \mathbf{v_2}, \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 2 & 2 \\ -1 & -3 & -8 \\ 1 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{-2} & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

A basis B_W of W is formed by the pivot columns of A. So $B_W = \{\mathbf{v}_1, \mathbf{v}_2\}$ and therefore $\dim(W) = 2$.

c) Find a basis of $\mathbb{R}^4$ by expanding the basis B_W you found in part **b)**. This means that you should find a basis of $\mathbb{R}^4$, that includes the basis vectors in B_W.

Solution: To extend B_W to a basis of $\mathbb{R}^4$ we have to find a simpler basis for Span$\{B_W\}$ = Span $\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Col}(B)$. We do this using "column reduction" of B. As we are unfamiliar with that, we use row reduction of B^T instead. This yields

$$B^T = \begin{bmatrix} 1 & 4 & -1 & 1 \\ 1 & 2 & -3 & 2 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 4 & -1 & 1 \\ 0 & \boxed{-2} & -2 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}.$$

Now $\tilde{B} = \{\mathbf{r}_1, \mathbf{r}_2\}$ is another basis for W. It can be completed to a basis of $\mathbb{R}^4$ using $\mathbf{e}_3$ and $\mathbf{e}_4$ in $\mathbb{R}^4$. As $\mathbf{v}_1$ and $\mathbf{v}_2$ also span W, a basis B_4 of $\mathbb{R}^4$ is

$$B_4 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

Note: This result can not be obtained from part **b)**. In general we might obtain different pivots in **b)** and **c)** as elementary row operations do change the column space.

(6) *(13 points)* Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases of a subspace W of dimension 2 in $\mathbb{R}^{123}$. We know that

$$\begin{array}{rrr} \mathbf{b}_1 = & \mathbf{c}_1 & -3\mathbf{c}_2 \\ \mathbf{b}_2 = & -2\mathbf{c}_1 & +4\mathbf{c}_2 \end{array}.$$

a) Write down the coordinate vectors $[\mathbf{b}_1]_{\mathcal{C}}$ and $[\mathbf{b}_2]_{\mathcal{C}}$ of the basis vectors in $\mathcal{B}$ with respect to the basis $\mathcal{C}$.

Solution: From the description above we obtain

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

b) Determine the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ that turns $\mathcal{B}$ coordinates into $\mathcal{C}$ coordinates.

Solution: The change-of-coordinates matrix is given by

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = [[\mathbf{b}_1]_{\mathcal{C}}, [\mathbf{b}_2]_{\mathcal{C}}] = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}.$$

c) Determine the change-of-coordinates matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ that turns $\mathcal{C}$ coordinates into $\mathcal{B}$ coordinates.

Solution: We have $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P^{-1}}$, so

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P^{-1}} = -\frac{1}{2}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}.$$

d) If $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$.

Solution: We have $[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\mathbf{x}]_{\mathcal{C}}$, so

$$[\mathbf{x}]_{\mathcal{B}} = -\frac{1}{2}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -9/2 \end{bmatrix}.$$

e) Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be another basis of W and $\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}$ be the change-of-coordinates matrix that turns $\mathcal{C}$ coordinates into $\mathcal{D}$ coordinates. Is $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P^{-1}} \cdot \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$? Justify your answer.

Solution: This is false. We have

$$\underset{\mathcal{D}\leftarrow\mathcal{C}}{P} \cdot \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{D}}. \text{ So } \underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P} \cdot \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}.$$

So the statement above is false.

(7) *(13 points)* Let

$$P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

a) Find a basis $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$, such that $P = \underset{V\leftarrow U}{P}$ is the change-of-coordinates matrix from U to $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: We know that $P_V = \underset{E\leftarrow V}{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is the change-of-coordinates matrix from V to the standard basis E. We also know that

$$\begin{aligned} \underset{V\leftarrow U}{P} &= \underset{V\leftarrow E}{P} \cdot \underset{E\leftarrow U}{P} = P_V^{-1} \cdot P_U. \text{ Therefore} \\ P_U &= [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3] = P_V \cdot \underset{V\leftarrow U}{P} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix}. \end{aligned}$$

b) Find a basis $W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, such that $P = \underset{W\leftarrow V}{P}$ is the change-of-coordinates matrix from V to W.

Solution: In a similar fashion we have

$$\begin{aligned} \underset{W\leftarrow V}{P} &= \underset{W\leftarrow E}{P} \cdot \underset{E\leftarrow V}{P} = P_W^{-1} \cdot P_V. \text{ So} \\ P_W &= [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = P_V \cdot \underset{W\leftarrow V}{P^{-1}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 3 \\ -1 & 2 & 4 \end{bmatrix}. \end{aligned}$$

We find $\underset{W\leftarrow V}{P^{-1}}$ by row reducing the augmented matrix $[\underset{W\leftarrow V}{P}|I_3]$ to the matrix $[I_3|\underset{W\leftarrow V}{P^{-1}}]$.

c) Find $\underset{W\leftarrow U}{P}$, the change-of-coordinates matrix from U to W.

Solution: We know that $\underset{W\leftarrow U}{P} = \underset{W\leftarrow V}{P} \cdot \underset{V\leftarrow U}{P}$, so

$$\underset{W\leftarrow U}{P} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}.$$

(8) *(12 points)* Suppose that A is a $m \times n$ matrix and B is a $n \times p$ matrix such that

$$A \cdot B = O,$$

where O is the zero matrix, i.e. the matrix where all entries are zeros.

a) What is the size of the matrix O, meaning how many rows and columns does it have? Explain your answer.

Solution: By the laws of matrix multiplication we have that O is an $m \times p$ matrix.

b) Show that $\mathrm{Col}(B)$ is a subset of $\mathrm{Nul}(A)$.

Solution: If $\mathbf{v} \in \mathrm{Col}(B)$, then $\mathbf{v} = B\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^p$. This implies

$$A\mathbf{v} = A(B\mathbf{x}) = AB\mathbf{x} = O\mathbf{x} = \mathbf{0}. \text{ So } A\mathbf{v} = \mathbf{0}.$$

So $\mathbf{v} \in \mathrm{Nul}(A)$. As $\mathbf{v}$ was chosen arbitrarily, we have that $\mathrm{Col}(B)$ is a subset of $\mathrm{Nul}(A)$.

c) Show that $n \geq \mathrm{Rank}(A) + \mathrm{Rank}(B)$.

Solution: The **Rank Theorem** tells us that

$$n = \mathrm{Rank}(A) + \dim\big(\mathrm{Nul}(A)\big).$$

We know from part **b)** that $\mathrm{Col}(B) \subset \mathrm{Nul}(A)$. Thus

$$\dim\big(\mathrm{Nul}(A)\big) \geq \dim\big(\mathrm{Col}(B)\big) = \mathrm{Rank}(B).$$

Therefore

$$n = \mathrm{Rank}(A) + \dim\big(\mathrm{Nul}(A)\big) \geq \mathrm{Rank}(A) + \mathrm{Rank}(B).$$