# Math 22 - Fall 2019
# Practice Exam 1

Your name: ________________________________

Section (please check the box): ☐ Section 1 (10 hour) ☐ Section 2 (2 hour)

INSTRUCTIONS

- Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer.
- It is fine to leave your answer in a form such as $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $\cos(\pi)$ or $(3-2)$), you should simplify it.
- You may use the last page for scrap paper.
- This is a closed book exam. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource.

GOOD LUCK!

(1) Please indicate whether the following statements are **TRUE** or **FALSE**. **Circle the correct answer**. You do not have to show your work, however thinking about the problem on a scrap paper is recommended.

1.) The linear system

$$\begin{aligned} 3x + 7y + z &= -1 \\ -x - y + z &= -3 \\ -2y - 2z &= \phantom{-}5 \end{aligned}$$

is consistent.

$\boxed{\textbf{TRUE}}$ **FALSE**

**Solution:** This is true. The augmented matrix of the system is

$$[A|\mathbf{b}] = \left[\begin{array}{ccc:c} 3 & 7 & 1 & -1 \\ -1 & -1 & 1 & -3 \\ 0 & -2 & -2 & 5 \end{array}\right] \to \left[\begin{array}{ccc:c} \boxed{2} & 0 & -4 & 11 \\ 0 & \boxed{2} & 2 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{ef}([A|\mathbf{b}]).$$

We see from the echelon form ef$([A|\mathbf{b}])$ of $[A|\mathbf{b}]$ that the system is consisten and has infinitely many solutions, as there is a non-pivot column which corresponds to a free variable.

2.) A vector $\mathbf{b}$ is in the span of the columns of a matrix $A$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

$\boxed{\textbf{TRUE}}$ **FALSE**

**Solution:** This is true. We have seen this in **Lecture 4, Theorem 3**.

3.) If there is a pivot in each row of a coefficient matrix, there are no free variables.

**TRUE** $\boxed{\textbf{FALSE}}$

**Solution:** This is false. If there is a pivot in every **column** of the coefficient matrix then there are no free variables.

4.) If there is a pivot in each row of an augmented matrix, the system is consistent.

**TRUE** $\boxed{\textbf{FALSE}}$

**Solution:** This is false. If there is a pivot in every row of the **coefficient matrix** then the system is consistent.

5.) A set of 3 vectors in $\mathbb{R}^4$ is always linearly independent.

**TRUE** $\boxed{\textbf{FALSE}}$

**Solution:** This is false. Take $\mathbf{v}_1 = \mathbf{0}$ and then choose any two other vectors $\mathbf{v}_2$ and $\mathbf{v}_3$, then the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

6.) A set of 6 vectors in $\mathbb{R}^7$ can never span $\mathbb{R}^7$.

$\boxed{\textbf{TRUE}}$ **FALSE**

**Solution:** This is true. Writing the 6 vectors as the columns of a matrix $A$, we have that $A$ is a $7 \times 6$ matrix. That the 6 vectors span $\mathbb{R}^7$ is equivalent to the statement that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^7$. This means that $A$ must have a pivot in every row. But the number of pivots is at most 6, so $A$ can never have a pivot in every row. So a set of 6 vectors in $\mathbb{R}^7$ can never span $\mathbb{R}^7$.

(2) Let

$$A = \begin{bmatrix} 3 & 0 & 3 & 0 \\ 1 & 2 & 3 & -4 \\ 2 & -1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix}$$

**a)** Solve the matrix equation $A\mathbf{x} = \mathbf{b}$, and write the solution in parametric vector form.

**Solution:** We row reduce the augmented matrix of the system:

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} 3 & 0 & 3 & 0 & 6 \\ 1 & 2 & 3 & -4 & 0 \\ 2 & -1 & 1 & 2 & 5 \end{array}\right] \overset{\frac{1}{3}R1}{\to} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 1 & 2 & 3 & -4 & 0 \\ 2 & -1 & 1 & 2 & 5 \end{array}\right] \overset{-R1+R2,-2R1+R3}{\to} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 2 & -4 & -2 \\ 0 & -1 & -1 & 2 & 1 \end{array}\right]$$

$$\overset{\frac{1}{2}R2}{\to} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -1 \\ 0 & -1 & -1 & 2 & 1 \end{array}\right] \overset{R2+R3}{\to} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 2 \\ 0 & \boxed{1} & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \text{ref}([A|\mathbf{b}]).$$

Then we write down the equations we obtain from the reduced echelon form ref$([A|\mathbf{b}])$:

$$\begin{aligned} x_1 &= 2 - x_3 \\ x_2 &= -1 - x_3 + 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

Then we rewrite these equations into vector form (by adding dummy zeros in the empty spaces):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

A nicer version with more emphasis on the vector equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}.$$

**b)** Without doing any row operations, write the solution set to the matrix equation $A\mathbf{x} = 5\mathbf{b}$.

**Solution:** By the linearity of the matrix vector multiplication we have: if $A\mathbf{z} = \mathbf{b}$, for some vector $\mathbf{z}$ then

$$5 \cdot A\mathbf{z} = A(5\mathbf{z}) = 5\mathbf{b}.$$

Furthermore we get all solutions $\mathbf{x}$, such that $A\mathbf{x} = 5\mathbf{b}$ this way: If $A\mathbf{p} = 5\mathbf{b}$, then it is easy to see that $A\left(\frac{1}{5} \cdot \mathbf{p}\right) = \mathbf{b}$ and so $\mathbf{p} = 5\left(\frac{1}{5} \cdot \mathbf{p}\right)$ or 5 times a vector $\mathbf{z}$ which satisfies $A\mathbf{z} = \mathbf{b}$. Therefore the set of solutions is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ 0 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \text{where } s, t \in \mathbb{R}. \quad (1)$$

Here the factor 5 can be eliminated in the parameter version, as $s, t \in \mathbb{R}$.
**Note:** Just the answer in the equation (1) is sufficient for this problem.

**c)** Write the solution to the homogenous equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

**Solution:** We row reduce the augmented matrix of the system:

$$[A|\mathbf{0}] = \left[\begin{array}{cccc|c} 3 & 0 & 3 & 0 & 0 \\ 1 & 2 & 3 & -4 & 0 \\ 2 & -1 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \text{ref}([A|\mathbf{0}]).$$

This involves the same steps as in part **a)**. We see that the fixed vector disappears and the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \text{ where } s, t \in \mathbb{R}.$$

**Note:** We could also argue with the theorem about the comparison between the solution to the homogeneous system and the nonhomogeneous system from **Lecture 4**.

**d)** Is the solution set of $A\mathbf{x} = \mathbf{0}$ a point, a line, a plane, a 3-dimensional space, or all of $\mathbb{R}^4$? Explain your answer.

**Solution:** The set of solutions is spanned by two linearly independent vectors. This means it is a plane.

(3) We want to find a polynomial $p(x)$ in the $xy$-plane that passes through the following points in the plane:

$$P_1 = (-2, 2)\,,\ P_2 = (-1, 3)\,,\ P_3 = (0, -2)\,,\ P_4 = (1, 0)\ \text{ and }\ P_5 = (2, 6).$$

We know that $p(x)$ is a polynomial of the following form.

$$p(x) = a + bx^2 + cx^3 + dx^5 + ex^6.$$

**a)** Write down the equations for the coefficients of $p$.

**Solution:** Each point $P_i = (x, y) = (x, p(x))$ gives an equation for the coefficients:

$$\begin{array}{rcccccc}
P_1 = (-2,2): & a & +b\cdot 2^2 & +c\cdot(-2^3) & +d\cdot(-2^5) & +e\cdot 2^6 & = 2\\
P_2 = (-1,3): & a & +b\cdot 1 & +c\cdot(-1) & +d\cdot(-1) & +e\cdot 1 & = 3\\
P_3 = (0,-2): & a & +b\cdot 0 & +c\cdot 0 & +d\cdot 0 & +e\cdot 0 & = -2\\
P_4 = (1,0): & a & +b\cdot 1 & +c\cdot 1 & +d\cdot 1 & +e\cdot 1 & = 0\\
P_5 = (2,6): & a & +b\cdot 2^2 & +c\cdot 2^3 & +d\cdot 2^5 & +e\cdot 2^6 & = 6
\end{array}$$

**b)** Write down the augmented matrix of the system of linear equations from **a)**.
You do **not** have to solve this system.

**Solution:** For the augmented matrix we obtain:

$$\left[\begin{array}{ccccc:c}
1 & 2^2 & -2^3 & -2^5 & 2^6 & 2\\
1 & 1 & -1 & -1 & 1 & 3\\
1 & 0 & 0 & 0 & 0 & -2\\
1 & 1 & 1 & 1 & 1 & 0\\
1 & 2^2 & 2^3 & 2^5 & 2^6 & 6
\end{array}\right]$$

(4) Let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ h \end{bmatrix} \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

**a)** Find all the real values $h$ for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans all of $\mathbb{R}^3$. Explain your answer.

**Solution:** Let $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ be the matrix whose columns are the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. Then the columns of $A$ span $\mathbb{R}^3$ if and only if the matrix equation $A\mathbf{x} = \mathbf{b}$ is always consistent. This is true if and only if $A$ has a pivot in every row. We row reduce $A$ to echelon form $U$ and see:

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & h \end{bmatrix} \overset{R1+R3}{\to} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 3+h \end{bmatrix} \overset{-2R2+R3}{\to} \begin{bmatrix} \boxed{-1} & 0 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 1+h \end{bmatrix} = U.$$

We see that $A$ has a pivot in every row if and only if $h \neq -1$. Therefore the vectors span $\mathbb{R}^3$ if and only if $h \neq -1$.

**b)** Let $h = 1$, and write $\mathbf{v}_4$ as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

**Solution:** The vector $\mathbf{v}_4$ is a linear combination of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{v}_4$$

has a solution. This is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}_4$, where the matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ and $\mathbf{x} = (x_1, x_2, x_3)$. This sytem of linear equations has the augmented matrix $[A|\mathbf{v}_4]$. Again we use the row reduction algorithm to solve this system of equations.

$$[A|\mathbf{v}_4] = \left[\begin{array}{ccc:c} -1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 0 \end{array}\right] \overset{R1+R3}{\to} \left[\begin{array}{ccc:c} -1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 4 & -2 \end{array}\right] \overset{-2R2+R3}{\to} \left[\begin{array}{ccc:c} \boxed{-1} & 0 & 3 & -2 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & \boxed{2} & -6 \end{array}\right]$$

$$\overset{\frac{1}{2}R3}{\to} \left[\begin{array}{ccc:c} \boxed{-1} & 0 & 3 & -2 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & \boxed{1} & -3 \end{array}\right] \overset{-R3+R2,-3R3+R1}{\to} \left[\begin{array}{ccc:c} \boxed{-1} & 0 & 0 & 7 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & -3 \end{array}\right] \overset{(-1)R1}{\to} \left[\begin{array}{ccc:c} \boxed{1} & 0 & 0 & -7 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & \boxed{1} & -3 \end{array}\right]$$

Hence $\mathbf{v}_4 = -7\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3$ and $\mathbf{v}_4$ can be written as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(5) Determine if each set is linearly dependent or linearly independent. Justify your answer.

**General solution:** In each case we can check the linear dependence equation, i.e if the zero vector $\mathbf{0}$ can be written as a non trivial linear combination of the given vectors. We can also check whether one of the vectors can be expressed as a linear combination of the others.

**a)** $\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$

**Solution:** In the case of two non zero vectors we know that they are linearly dependent if and only if one is the multiple of the other, or

$$c \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

However, this can not be true, so these two vectors are linearly independent.

**b)** $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \end{bmatrix} \right\}$

**Solution:** We check whether there are numbers $x_1, x_2$ and $x_3$, not all zero, such that

$$x_1 \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, the last row gives us $2x_3 = 0$, so $x_3 = 0$. This implies $x_1 = 0$ in the third row. Finally this implies $x_2 = 0$ in the second row. So the only solution is $x_1 = x_2 = x_3 = 0$ and the three vectors are linearly independent.

**c)** $\left\{ \begin{bmatrix} 10 \\ -4 \\ -1 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 9 \\ -3 \end{bmatrix}, \begin{bmatrix} -7 \\ -10 \\ -2 \\ -9 \end{bmatrix}, \begin{bmatrix} 7 \\ -8 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 1 \\ 5 \end{bmatrix} \right\}.$

**Solution:** When we have $5$ vectors in $\mathbb{R}^4$ then they must be linearly dependent. To see that we write the vectors into the columns of a $4 \times 5$ matrix $A$. The linear dependence equation then translates to finding a nontrivial vector $\mathbf{x}$ such that

$$A\mathbf{x} = \mathbf{0}.$$

As $A$ has only $4$ rows, it can only have $4$ pivot columns. Hence $A$ has a non-pivot column and there is at least one free variable for $\mathbf{x}$. So there are non-trivial solutions and the vectors can not be linearly independent.

**Note:** A shorter answer is possible.

(6) Let $T$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2}x_1 + \frac{1}{2}x_2 \\ x_1 - x_2 \\ x_2 \end{bmatrix}.$$

**a)** Find the standard matrix for $T$.

**Solution:**

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2}x_1 + \frac{1}{2}x_2 \\ x_1 - x_2 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So the standard matrix $A$, such that $A\mathbf{x} = T(\mathbf{x})$ is $A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$.

**Alternative:** We find that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}. \text{ Then } A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & -1 \\ 0 & 1 \end{bmatrix},$$

by a **Theorem** from the lecture.

**b)** Find a vector $\mathbf{x}$ whose image under $T$ is $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$.

**Solution:** We have to find a vector $\mathbf{x}$, such that $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \mathbf{b}$. To this end we solve the system of linear equations with augmented matrix $[A|\mathbf{b}]$:

$$\left[\begin{array}{cc|c} \frac{3}{2} & \frac{1}{2} & 5 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{array}\right] \overset{R1 \leftrightarrow R2}{\to} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ \frac{3}{2} & \frac{1}{2} & 5 \\ 0 & 1 & 1 \end{array}\right] \overset{-\frac{3}{2}R1+R2}{\to} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{array}\right] \overset{\frac{1}{2}R2}{\to} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$\overset{-R2+R3}{\to} \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \overset{-R2+R1}{\to} \left[\begin{array}{cc|c} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{array}\right]. \text{ We obtain that } \boxed{\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}.$$

**c)** Is $T$ one-to-one? Justify your answer.

**Solution:** Yes, because the columns of the standard matrix are linearly independent.
**Alternative:** Yes, because $A$ has a pivot in every column.
For both statements see **Lecture 8**.

(7) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first performs a horizontal shear that transforms $\mathbf{e}_2$ into $\mathbf{e}_2 - 2\mathbf{e}_1$ (leaving $\mathbf{e}_1$ unchanged) and then reflects points through the line $x_2 = -x_1$.

Find the standard matrix of $T$.

**Solution:** The composition of a two linear transformations is a linear transformation. We know that $T = R \circ H$, where $H$ denotes the horizontal shear and $R$ the reflection.

To find the standard matrix $A$, such that $A\mathbf{x} = T(\mathbf{x})$, we have to find the images $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ of the unit vectors. We have:

$$\begin{aligned} H(\mathbf{e}_1) &= \mathbf{e}_1 \text{ and } T(\mathbf{e}_1) = R(H(\mathbf{e}_1)) = R(\mathbf{e}_1) = R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \\ H(\mathbf{e}_2) &= \mathbf{e}_2 - 2\mathbf{e}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = R(H(\mathbf{e}_1) = R\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}. \end{aligned}$$

By **Lecture 8** the standard matrix $A$ is

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}.$$

(8) Suppose that $T$ is a one-to-one linear transformation and that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent vectors in $\mathbb{R}^n$. Prove that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\}$ is also linearly independent.

**Solution:** As $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent vectors we know that

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \ldots + d_p\mathbf{v}_p = \mathbf{0} \text{ implies } d_1 = d_2 = \ldots = d_p = 0. \quad (1)$$

We want to show that

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_pT(\mathbf{v}_p) = \mathbf{0} \text{ implies } c_1 = c_2 = \ldots = c_p = 0.$$

However, by linearity

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \ldots + c_pT(\mathbf{v}_p) = \mathbf{0} \text{ implies } T(\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p}_{=\mathbf{w}}) = \mathbf{0}.$$

Hence the vector $\mathbf{w} = \mathbf{0}$ as the map $T$ is one-to-one. So the only vector whose image is the zero vector is the zero vector $\mathbf{0}$ in $\mathbb{R}^n$. So we conclude that

$$\mathbf{0} = \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p.$$

But this implies that $c_1 = c_2 = \ldots = c_p = \mathbf{0}$ by equation (1). This proves our claim.

*(This page is intentionally left blank in case you need extra space for any of the problems.* **If you use this page for a particular problem, it is essential that you make a note on the page where the problem appears, indicating that your work is continued here.)**