Math 22 - Fall 2019
Practice Exam 2

Your name: ______________________________

Section (please check the box): ☐ Section 1 (10 hour) ☐ Section 2 (2 hour)

INSTRUCTIONS

- Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer.
- It is fine to leave your answer in a form such as $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $\cos(\pi)$ or $(3-2)$), you should simplify it.
- You may use the last page for scrap paper.
- This is a closed book exam. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource.

GOOD LUCK!

(1) Please indicate whether the following statements are **TRUE** or **FALSE**.
Circle the correct answer. You do not have to show your work, however thinking about the problem on a scrap paper is recommended.

1.) If $\dim(V) = p$, and if S is a linearly dependent subset of V, then S contains more than p vectors.

TRUE **FALSE**

2.) If a 6×4 matrix A has linearly independent columns, then the reduced row echelon form of A contains two zero rows.

TRUE **FALSE**

3.) There exists a 3×5 matrix whose column space has dimension 4.

TRUE **FALSE**

4.) There exists a 3×3 matrix A such that $\dim(\text{Nul}(A)) = \text{Rank}(A)$.

TRUE **FALSE**

5.) For any two $n \times n$ matrices A and B, we have $\det(AB) = \det(B^T A)$.

TRUE **FALSE**

6.) Let P be a subset of $\mathbb{P}_2$, the polynomials of degree at most 2, defined by

$$P = \{\mathbf{p}(t) \text{ in } \mathbb{P}_2 : \mathbf{p}(1) = 2\}\,.$$

Then P is a subspace of $\mathbb{P}_2$.

TRUE **FALSE**

7.) Let $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ be the $n \times n$ change of basis matrix that goes from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates. Then, the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ span $\mathbb{R}^n$.

TRUE **FALSE**

(2) Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

For each of the following matrix operations, indicate whether the operation is defined. If an expression is undefined, explain why. If an expression is defined, evaluate it.

a) $BA + C$.

b) BC.

c) $B + 3I_2$

(3) **a)** Given that

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -6$$

find the value of

$$\det \begin{pmatrix} d & e & f \\ 4a & 4b & 4c \\ g-3d & h-3e & i-3f \end{pmatrix}.$$

b) Suppose that A, B and C are 2×2 matrices with

$$\det(A) = 9, \det(C) = -\frac{1}{2} \text{ and } \det\left(\frac{1}{3}A^T BC^2\right) = -2.$$

Find $\det(B)$, if possible, or explain why you cannot.

(4) Assume that A is row-equivalent to B.

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) Find a basis for $\mathrm{Col}(A)$.

b) Find a basis for $\mathrm{Nul}(A)$.

c) Find a basis for $\mathrm{Row}(A)$.

(5) Let $W = \text{Span}(S)$, where $S = \left\{ \begin{bmatrix} 3 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \right\}$ is a set of vectors in $\mathbb{R}^4$.

a) Find a subset R of S which is a basis for W. What is the dimension of W?

b) Give an example of a vector $\mathbf{v}$ which is in $\mathbb{R}^4$ but is not in W. Justify your answer.

c) Find a basis B for $\mathbb{R}^4$ by expanding the basis R you found in part **a)**. Explain why B is a basis for $\mathbb{R}^4$.

(6) Let $\mathbf{p}_1(t) = 1, \mathbf{p}_2(t) = t+1, \mathbf{p}_3(t) = (t+1)^2, \mathbf{p}_4(t) = (t+1)^3$ be four polynomials in $\mathbb{P}_3$. Let

$$B = \{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t), \mathbf{p}_4(t)\}, \text{ and let } E = \{1, t, t^2, t^3\}$$

be the standard basis of $\mathbb{P}_3$.

a) Determine the coordinates of the vectors in B with respect to the basis E.

b) Determine the coordinates of the vectors in E with respect to the basis B.

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c) Suppose $\left[\mathbf{q}(t)\right]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$. Find $\mathbf{q}(t)$.

d) Let $\mathbf{p}_1(t) = 1, \mathbf{p}_2(t) = t+1, \mathbf{p}_3(t) = (t+1)^2, \ldots, \mathbf{p}_{n+1}(t) = (t+1)^n$ be polynomials in $\mathbb{P}_n$. Show that the set

$$\{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t), \ldots, \mathbf{p}_{n+1}(t)\}$$

forms a basis of $\mathbb{P}_n$. (Hint: think about dimension.)

(7) Let V be the vector space of 2×2 upper triangular matrices, so that

$$V = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Let

$$\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

be the standard basis of V, and consider the alternate bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

(a) Find $\underset{\mathcal{E} \leftarrow \mathcal{B}}{P}$.

(b) Find $\underset{\mathcal{C} \leftarrow \mathcal{E}}{P}$.

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(c) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

(d) If $\left[\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, find $\left[\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right]_{\mathcal{C}}$.

(8) Let A be a $m \times n$ matrix and C be a $n \times m$ matrix, such that

$$AC = I_m.$$

a) Show that the map $T : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x}$ is onto.

b) Show that the map $S : \mathbb{R}^m \to \mathbb{R}^n, \mathbf{x} \mapsto S(\mathbf{x}) = C\mathbf{x}$ is one-to-one.

c) Is $m \geq n$ or $n \geq m$? Justify your answer.

d) Let $R : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto R(\mathbf{x}) = B\mathbf{x}$ be a linear map such that R is onto. Show that there is a matrix D, such that

$$BD = I_m$$

(This page is intentionally left blank in case you need extra space for any of the problems. **If you use this page for a particular problem, it is essential that you make a note on the page where the problem appears, indicating that your work is continued here.)**