Math 22 - Fall 2019
Practice Exam 3

Your name: ______________________________

Section (please check the box): ☐ Section 1 (10 hour) ☐ Section 2 (2 hour)

INSTRUCTIONS

- Except on clearly indicated short answer problems, you must explain what you are doing, and show your work. You will be *graded on your work*, not just on your answer.
- It is fine to leave your answer in a form such as $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $\cos(\pi)$ or $(3-2)$), you should simplify it.
- You may use the last page for scrap paper.
- This is a closed book exam. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource.

GOOD LUCK!

(1) Please indicate whether the following statements are **TRUE** or **FALSE**.
Circle the correct answer. You do not have to show your work, however thinking about the problem on a scrap paper is recommended.

1.) If $\dim(V) = p$, and if S is a linearly dependent subset of V, then S contains more than p vectors.

TRUE **FALSE**

2.) If A is a 2×4 matrix, then the corresponding map $T(\mathbf{x}) = A\mathbf{x}$ is always onto.

TRUE **FALSE**

3.) There exists a 3×5 matrix whose row space has dimension 2.

TRUE **FALSE**

4.) Let U and Q be two orthogonal **matrices**. Then UQ is also an orthogonal matrix.

TRUE **FALSE**

5.) Let U be an orthogonal **matrix**. Then the corresponding map $S(\mathbf{x}) = U\mathbf{x}$ preserves angles. This means that the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $S(\mathbf{x})$ and $S(\mathbf{y})$.

TRUE **FALSE**

6.) For any two 4×4 matrices A and B, we have $\det(3AB) = 3^2 \cdot \det(B)\det(A)$.

TRUE **FALSE**

7.) Let A be an $m \times n$ matrix. Then $\dim(\mathrm{Col}(A)) = \dim(Col(A^T))$.

TRUE **FALSE**

8.) Let $\mathbf{v}$ be an eigenvector of the matrix A with eigenvalue λ and also of the matrix B with eigenvalue μ. Then $\mathbf{v}$ is an eigenvector of the matrix AB.

TRUE **FALSE**

9.) Every stochastic matrix P has a unique steady state vector $\mathbf{q}$.

TRUE **FALSE**

(2) Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

For each of the following matrix operations, indicate whether the operation is defined. If an expression is undefined, explain why. If an expression is defined, evaluate it.

a) $BA + C$.

b) BC.

c) $B + 3I_2$

(3) Find the inverse of the following matrix or show that it does not exist.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}.$$

(4) Consider the three polynomials in $\mathbb{P}_2$:

$$p_1(t) = 1 + t^2 \ , \ p_2(t) = t - 3t^2 \ , \ p_3(t) = 1 + t - 3t^2 .$$

a) Use coordinate vectors to show that these polynomials form a basis B of $\mathbb{P}_2$.

b) Find $q \in \mathbb{P}_2$, such that $[q]_B = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

(5) Let H and K be subspaces of a vector space V. Show that the intersection $H \cap K$ is a subspace of V. Then give a counterexample in $\mathbb{R}^2$ that the union $H \cup K$ is not, in general, a subspace.

(6) Let

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

a) Find a basis $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$, such that $P = \underset{V \leftarrow U}{P}$ is the change-of-coordinates matrix from U to $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b) Find a basis $W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, such that $P = \underset{W \leftarrow V}{P}$ is the change-of-coordinates matrix from V to W.

c) Find $\underset{W \leftarrow U}{P}$, the change-of-coordinates matrix from U to W.

(7) Let $W = \text{Span}\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\4\\4\end{bmatrix}\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

a) Find the cosine of the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) Find the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$.

c) Find $W^{\perp}$ and determine the dimension of W and $W^{\perp}$.

(8) Find the equation of the line $y = f(x) = \beta_0 + \beta_1 x$ of the least-squares line that fits best the data points

$$P_1 = (0, 1)\ ,\ \ P_2 = (1, 1)\ ,\ \ P_3 = (2, 2)\ \text{ and }\ P_4 = (3, 2).$$

(9) Let

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

Let W be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$.

a) Use the Gram-Schmidt process to find an orthogonal basis of W.

b) Find the orthogonal projection of $\mathbf{y}$ onto W.

c) Find the minimum of $\|\mathbf{y} - \mathbf{w}\|$ where $\mathbf{w} \in W$.

(10) Let A be the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) Find the eigenvalues of A and determine their multiplicity.

b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

c) Diagonalize the matrix A.

d) Calculate A^7.

(11) Let A be a $n \times n$ matrix that is **diagonalizable**.

a) Give the definition of a diagonalizable matrix.

b) Show that A^T, the transpose of A, has exactly the same eigenvalues as A.

c) Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis of eigenvectors for A. Find a basis of eigenvectors for A^T.

(This page is intentionally left blank in case you need extra space for any of the problems. **If you use this page for a particular problem, it is essential that you make a note on the page where the problem appears, indicating that your work is continued here.)**