Math 22
Homework # 3

Write careful solutions for the homework that demonstrates a command of what you have learned on week #3. Do not carry out computations without telling the reader why you are doing the computation. If you say something is true provide a short explanation using definitions or Theorems. Hand-in something that you can feel proud of.

1. Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$.
   (a) Find polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ that span the kernel of T.
   (b) Describe the range of T.

2. Find the standard matrix for the following linear transformations.
   (a) The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that satisfy $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$.
   (b) The linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points (about the origin) through the angle $-\pi/4$ (clockwise).
   (c) The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = \begin{bmatrix} x_1 - 5x_2 + 4x_3, x_2 - 6x_3 \end{bmatrix}$

3. Explain why if the columns of B are linearly dependent, then so are the columns of the matrix AB.

4. Suppose that A is an $n \times n$ matrix and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^n$. Explain why A has to be invertible.