PROOF PROBLEMS

MATH 22

(1) Prove that if λ is an eigenvalue of AB then λ is an eigenvalue of BA.

(2) Suppose that $T : V \to W$ is an isomorphism of V onto W.
   (a) Show that H is a subspace of V if and only if $T(H) := \{T(\mathbf{v}) \in W \mid \mathbf{v} \in H\}$ is a subspace of W.
   (b) Let H be a subspace of V. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a basis for H if and only if $\{T(\mathbf{v}_1, T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p\}$ is a basis for $T(H)$.

(3) Let A_n be the $n \times n$ matrix that has zeros on its main diagonal and all other entries equal to -1. Find the determinant of A_n. [HINT: Add all rows (except the last) to the last row, then factor our a constant. Try $n = 3$.]

(4) Suppose that u is a unit vector in $\mathbb{R}^n$, so that $u^T u = 1$. Let $H = I - 2uu^T$. H is an $n \times n$ symmetric matrix.
   (a) Show that $H^2 = I$. What can you say about H in this case?
   (b) One eigenvector of H is u. Find the corresponding eigenvalue.
   (c) If v is perpendicular to u, show that v is an eigenvector of H and find its eigenvalue.

(5) Let A be an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{v}$. Use mathematical induction to show that λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{v}$ for all integers $n \geq 1$.

Date: August 15, 2012.