Lecture 13

Math 22 Summer 2017
July 17, 2017

Reminders/Announcements

- Midterm 1 Wednesday, July 19, Kemeny 008, 6pm-8pm
  - Please respond to email if you have a conflict with this time!
  - Kate is moving her study group to Wednesday 3pm - 4:30pm in the usual place (Berry 370). She is not having study group this Sunday.
- Last scheduled x-hour meets tomorrow.
- Thursday office hours are on Tuesday this week due to: `https://wiki.sagemath.org/days87`
- HW4 will be posted later today and due Friday as usual.
- Any other questions (content or otherwise) please feel free to email me.

Just for today

- §4.1 Two examples we didn't get to last time
- §4.2 Null and column spaces corresponding to linear maps

§4.1 Proving a set is a subspace

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How can we use the previous theorem to show H is a subspace? Well,

$$H = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -7 \end{bmatrix} \right\}.$$

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How do we show H is not a subspace?

§4.2 Definition of null space

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Show $\mathbf{0} \in \operatorname{Nul} A$. Show $\operatorname{Nul} A$ closed under addition.

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Proof.

Show $\mathbf{0} \in \operatorname{Nul} A$. Show $\operatorname{Nul} A$ closed under addition. Show $\operatorname{Nul} A$ closed under scalar multiplication. □

§4.2 An explicit description for Nul A

Let

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

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Now write out the parametric vector form of the solutions to $A\mathbf{x} = \mathbf{0}$. Can you see how this yields a spanning set? Is this set linearly independent?

§4.2 Null spaces as kernels of linear maps

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How does $\ker T$ relate to $\operatorname{Nul} A$?

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What vector space is $\mathrm{Col}\, A$ a subspace of?
When is $\mathrm{Col}\, A = \mathbb{R}^m$?

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Let $T : V \to W$ be a linear map of vector spaces. The **image** or **range** of T is the set of all vectors $\mathbf{b} \in W$ such that there exists $\mathbf{x} \in V$ and $T(\mathbf{x}) = \mathbf{b}$.

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How does $\operatorname{img} T$ relate to $\operatorname{Col} A$?

§4.2 Classwork

Let

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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1. $\operatorname{Nul} A \subseteq \mathbb{R}^k$ for what k? $\operatorname{Col} A \subseteq \mathbb{R}^k$ for what k?
2. Find a nonzero vector in $\operatorname{Nul} A$. Find a nonzero vector in $\operatorname{Col} A$.

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1. $\mathrm{Nul}\, A \subseteq \mathbb{R}^k$ for what k? $\mathrm{Col}\, A \subseteq \mathbb{R}^k$ for what k?
2. Find a nonzero vector in $\mathrm{Nul}\, A$. Find a nonzero vector in $\mathrm{Col}\, A$.
3. Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Is $\mathbf{u} \in \mathrm{Nul}\, A$? Is $\mathbf{u} \in \mathrm{Col}\, A$? Is $\mathbf{v} \in \mathrm{Nul}\, A$? Is $\mathbf{v} \in \mathrm{Col}\, A$?