Lecture 16

Math 22 Summer 2017
July 24, 2017

Just for today

- §4.4 Finish up
- §4.5 Dimension

The matrix of a linear transformation revisited

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We define **the matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$, denoted ${}_{\mathcal{C}}[T]_{\mathcal{B}}$** by

$$ {}_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{b}_1)]_{\mathcal{C}} & [T(\mathbf{b}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{b}_n)]_{\mathcal{C}} \end{bmatrix}. $$

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How does this relate to coordinate vectors?

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Thus, if $\mathcal{B}$ and $\mathcal{C}$ are bases of the *same* vector space V, We can use the matrix of a linear transformation to write coordinate vectors with respect to different bases (i.e. to change coordinates). The key property of ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ is that

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Thus, if $\mathcal{B}$ and $\mathcal{C}$ are bases of the *same* vector space V, then we can relate the coordinate vectors of any element of $\mathbf{x}$ using the identity linear transformation $\mathrm{id} : V \to V$ in the following way.

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The matrix ${}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{B}}$ is called the **change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**. Let's see how this works in our classwork example (back page)! `https://math.dartmouth.edu/~m22x17/section2lectures/classwork15.pdf`

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What is the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$?

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What is the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$? Well,

$$ {}_{\mathcal{C}}[T]_{\mathcal{B}} = \Big[\, [T(1)]_{\mathcal{C}} \ [T(t)]_{\mathcal{C}} \ [T(t^2)]_{\mathcal{C}} \ [T(t^3)]_{\mathcal{C}} \,\Big] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}. $$

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$$T(2 + 3t + 4t^2 + 5t^3).$$

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and $T(2 + 3t + 4t^2 + 5t^3) = 3 + 8t + 15t^2$.

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Proof.

Map to coordinates and use the same fact about $\mathbb{R}^n$ to get a dependence relation. $\square$

§4.5 Theorem 10

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$$\#\mathcal{B} \leq \#\mathcal{B}'$$
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We denote the dimension of V by $\dim V$.

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Examples?

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Examples? What about subspaces?

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Note that this proof works in the infinite-dimensional case as well, but requires Zorn's Lemma.

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Proof.

Corollary of previous theorem and the spanning set theorem. □

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What's the proof?

§4.5 Classwork

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Find the dimension of the subspace H of $\mathbb{R}^3$ defined by

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{array}{r} x + y = 0 \\ y + z = 0 \\ x - z = 0 \end{array} \right\}.$$

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Solution:

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Solution: First why is H a subspace? Because $H = \operatorname{Nul} A$ for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

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Now, how does this tell us the dimension of H?