Lecture 18

Math 22 Summer 2017
July 28, 2017

Just for today

- §4.6 Rank and the IMT
- §5.1 Eigenvectors and eigenvalues

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In the situation where $A\mathbf{x} = \lambda\mathbf{x}$ we say that $\mathbf{x}$ is an eigenvector corresponding to λ.

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This is our first example of an *eigenspace* which we now define...

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What are the eigenspaces for $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$?

§5.1 Classwork

1. Is $\mathbf{x} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$ an eigenvector for $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?
2. Is $\lambda = 3$ an eigenvalue of A?
3. Let $B = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 9$. Find a basis for the eigenspace corresponding to $\lambda = 2$. What is the dimension of this space?
4. Using A and $\mathbf{x}$ defined above, compute $A^2\mathbf{x}, A^3\mathbf{x}, \ldots, A^k\mathbf{x}$.
5. Using A defined above, write $A - \lambda I_2$ as a matrix (for arbitrary λ). Now compute $\det(A - \lambda I_2)$. For what values of λ is $\det(A - \lambda I_2) = 0$?

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Let $A = (a_{ij})$. λ is an eigenvalue of A if and only if the null space of $A - \lambda I_n$ contains a nonzero vector. Write out $A - \lambda I_n$ under the assumption that A is triangular to show that $(A - \lambda I_n)\mathbf{x} = \mathbf{0}$ has a free variable precisely when $\lambda = a_{kk}$ for some $k \leq n$. □

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Suppose A has $\lambda = 0$ as an eigenvalue? What can you say about the invertibility of A?

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We will prove this by contradiction.

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How can we use the boxed equations to get a contradiction? □

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How can we use this to simplify the computation of $\mathbf{x}_k$ for large values of k?