Lecture X01

Math 22 Summer 2017 Section 2
June 27, 2017

Introduction to proofs

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We will start with some simple examples...

Definition

A **mammal** is a warm-blooded animal.

Proof by example

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Sometimes the theorem just requires us to exhibit a specific example.

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Theorem

Mammals exist.

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Theorem

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Proof.

At least one human exists. Humans are mammals. □

Direct proof

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Theorem

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Proof.

Humans are mammals. Mammals are warm-blooded. □

Proof by contradiction

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Theorem

Human x is warm-blooded.

Proof.

Assume x is cold-blooded. Then a mammal would be cold-blooded which is impossible (a contradiction) by the definition of mammal. □

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If x cold-blooded, then x is not a human.

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Let A, B be the following statements:

$$\begin{aligned} \text{A}: &\ x \text{ is cold-blooded.} \\ \text{B}: &\ x \text{ is not a human.} \end{aligned}$$

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Let A, B be statements.

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Proof.

Let A, B be the following statements:

$$\text{A}: \; x \text{ is cold-blooded.}$$
$$\text{B}: \; x \text{ is not a human.}$$

By the previous theorem, we know that $\neg B \implies \neg A$, so the current theorem follows by contrapositive. □

Equality of sets

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Theorem

$$\underbrace{\text{Span}\{\mathbf{u}, \mathbf{v}\}}_{A} = \underbrace{\text{Span}\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}}_{B}.$$

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$B \subseteq A$: Let $c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{u}-\mathbf{v})$ be an arbitrary element of B.

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Proof.

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Then $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = (c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} \in A$.

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$A \subseteq B$: Let $c_1\mathbf{u} + c_2\mathbf{v} \in A$ with $c_1, c_2 \in \mathbb{R}$.

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$$\underbrace{\text{Span}\{\mathbf{u}, \mathbf{v}\}}_{A} = \underbrace{\text{Span}\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}}_{B}.$$

Proof.

$B \subseteq A$: Let $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v})$ be an arbitrary element of B.
Then $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = (c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} \in A$.
$A \subseteq B$: Let $c_1\mathbf{u} + c_2\mathbf{v} \in A$ with $c_1, c_2 \in \mathbb{R}$. Then

$$c_1\mathbf{u} + c_2\mathbf{v} = \frac{c_1 + c_2}{2}(\mathbf{u} + \mathbf{v}) + \frac{c_1 - c_2}{2}(\mathbf{u} - \mathbf{v}) \in B.$$

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Let A and B be sets.
To show $A = B$ it is suffices to show $A \subseteq B$ and $B \subseteq A$.

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$$\underbrace{\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}}_{A} = \underbrace{\mathrm{Span}\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}}_{B}.$$

Proof.

$B \subseteq A$: Let $c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{u}-\mathbf{v})$ be an arbitrary element of B.
Then $c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{u}-\mathbf{v}) = (c_1+c_2)\mathbf{u} + (c_1-c_2)\mathbf{v} \in A$.
$A \subseteq B$: Let $c_1\mathbf{u} + c_2\mathbf{v} \in A$ with $c_1, c_2 \in \mathbb{R}$. Then

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Since $A \subseteq B$ and $B \subseteq A$, we conclude that $A = B$. □

1×1 linear systems

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Consider the 1×1 linear system: $ax = b$, $a, b \in \mathbb{R}$.

1×1 linear systems

Consider the 1×1 linear system: $ax = b$, $a, b \in \mathbb{R}$. For each of the following claims prove the claim, give a counterexample, or prove the claim is false. Compare your arguments with you neighbors and see if you believe each other!

1×1 linear systems

Claim

If $b = 0$, then $ax = b$ is consistent for any a.

1×1 linear systems

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Proof.

By example: We exhibit a solution (namely $x = 0$) that works for every a. □

1×1 linear systems

Claim

Let $a, b \in \mathbb{R}$*. Then* $ax = b$ *has a solution.*

1×1 linear systems

Claim

Let $a, b \in \mathbb{R}$. Then $ax = b$ has a solution.

Proof.

The claim is false. $a = 0, b = 1$ is a counterexample. We could also take b to be anything nonzero. □

1×1 linear systems

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Since $a \neq 0$, we have the solution $x = b/a$. This proves the system is consistent.

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If $a \neq 0$, then for any b, the system $ax = b$ is consistent and has a unique solution.

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Proof.

Since $a \neq 0$, we have the solution $x = b/a$. This proves the system is consistent. Assume there is another solution y with $ay = b$. Then $ax = ay$ since they are both equal to b.

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Proof.

Since $a \neq 0$, we have the solution $x = b/a$. This proves the system is consistent. Assume there is another solution y with $ay = b$. Then $ax = ay$ since they are both equal to b. Thus $a(x - y) = 0$.

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Since $a \neq 0$, we have the solution $x = b/a$. This proves the system is consistent. Assume there is another solution y with $ay = b$. Then $ax = ay$ since they are both equal to b. Thus $a(x - y) = 0$. Now, since $a \neq 0$, we must have $x = y$.

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Since $a \neq 0$, we have the solution $x = b/a$. This proves the system is consistent. Assume there is another solution y with $ay = b$. Then $ax = ay$ since they are both equal to b. Thus $a(x - y) = 0$. Now, since $a \neq 0$, we must have $x = y$. So the solution is unique. $\square$

1×1 linear systems

Claim

There is some choice of $a, b \in \mathbb{R}$ *so that* $ax = b$ *has exactly 2 solutions.*

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Suppose there are 2 distinct solutions $x, y, x \neq y$.

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Suppose there are 2 distinct solutions $x, y, x \neq y$. Then $a(x - y) = 0$. Since $x \neq y$ we must have $a = 0$.

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Claim

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Proof.

Suppose there are 2 distinct solutions $x, y, x \neq y$. Then $a(x - y) = 0$. Since $x \neq y$ we must have $a = 0$. Since the system is consistent (we assumed we had solutions) we must have $b = 0$.

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Claim

There is some choice of $a, b \in \mathbb{R}$ so that $ax = b$ has exactly 2 solutions.

Proof.

Suppose there are 2 distinct solutions $x, y, x \neq y$. Then $a(x - y) = 0$. Since $x \neq y$ we must have $a = 0$. Since the system is consistent (we assumed we had solutions) we must have $b = 0$. Does this prove or disprove the claim? □

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Claim

If $b = 0$, then $ax = b$ always has a unique solution.

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Claim

If $b = 0$, then $ax = b$ always has a unique solution.

Proof.

If $a = 0$, then any x is a solution. □