Math 23 Diff Eq: Review of 2-by-2 matrices

Alex Barnett, October 15, 2007

You will need to know or pick up these basics for this course (we postpone eigenvalues and eigenvectors until later).

A pair of simultaneous linear equations for two unknowns x_1, x_2 can generally be written

$$\left.\begin{array}{rcl} ax_1 + bx_2 &=& y_1 \\ cx_1 + dx_2 &=& y_2 \end{array}\right\} \tag{1}$$

The linear algebra way to write the same is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

which can be summarized by

$$M\mathbf{x} = \mathbf{y} \tag{2}$$

where M is an order-2 square matrix and $\mathbf{x}$ and $\mathbf{y}$ are column vectors. Make sure you understand how the matrix 'hits' the $\mathbf{x}$ vector to give the LHS terms in Eq. (1): the product $M\mathbf{x}$ means literally build a vector linear combination given by x_1 amount of the first column of M plus x_2 amount of the second column.

Matrices multiply as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

This is just the same as the matrix-vector product, for each column vector of the matrix on the right (the one getting 'hit' from the left). Alternatively, the $(i,j)^{th}$ entry in the product is given by the dot product of the i^{th} row of the left matrix with the j^{th} column of the right matrix. Draw a picture so you get this. In contrast to scalars, matrix multiplication is not *commutative*, *i.e.* in general $AB \neq BA$. Order matters!

The determinant is $\det M := ad - bc$ for the above matrix, also written $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. If $\det M = 0$ we say M is 'singular'. There are more complicated expressions for det of 3-by-3 and higher. If A and B are same-sized square matrices then $\det(AB) = \det A \cdot \det B$.

All nonsingular matrices are invertible, that is a same-sized matrix M^{-1} exists such that $MM^{-1} = I$ and $M^{-1}M = I$. Here $I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix, which has ones only on the 'diagonal'. Singular matrices are not invertible. The formula for the 2-by-2 inverse is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{3}$$

Notice that the factor out front is $1/(\det M)$.

It's a beautiful result of linear algebra that (1) is uniquely solvable when $\det M \neq 0$. One way to get the solution is to eliminate a variable from the simultaneous equations and solve the high-school way. A more elegant way is to multiply (2) from the left by M^{-1} to give

$$\mathbf{x} = M^{-1}\mathbf{y} \tag{4}$$

an explicit solution showing the elegance of matrix algebra.

You can 'transpose' M by reflecting along the diagonal to give

$$M^T := \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Note that 2-by-2 matrices can represent linear (homogeneous) transformations on the plane, *i.e.* stretching, shear, rotation...

All of the above (suitably generalized) applies to n-by-n matrices too!

Exercises

1. Say $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $\det M$ and M^{-1}. Solve $M\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = [6 \ -1]^T$.
2. if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = A^T$, compute AB and BA. Are they equal?
3. If $\begin{bmatrix} a & 2 \\ 3 & 4 \end{bmatrix}$ is singular, what is a?
4. You may also have functions (*e.g.* of t) as matrix entries. Find $\begin{vmatrix} e^t & e^{-4t} \\ e^{3t} & e^{-t} \end{vmatrix}$. Find $\begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix}$ and simplify.
5. Verify that M and M^{-1} given by (3) multiply to give I, in either order.
6. Verify that the general solution to (1) given by elimination matches that using (4).
7. If A is singular, prove using the above that AB is not invertible for any B.
8. Prove using the above that $\det M^{-1} = (\det M)^{-1}$.

Answers:

1. -2, $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, $\mathbf{x} = [-13 \ 19/2]^T$.
2. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.
3. $a = 3/2$.
4. $1 - e^{-t}$, and 1.
5. it works
6. $x_1 = (dy_1 - by_2)/(ad - bc)$, $x_2 = (-cy_1 + ay_2)/(ad - bc)$
7. $\det AB = 0 \cdot \det B = 0$ always.
8. $1 = \det I = \det(MM^{-1}) = \det M \cdot \det M^{-1}$ now rearrange to give result.