Math 23 Diff Eq: Midterm Fall 2007

2 hours, 60 points. Answer all five questions, giving as much explanation as you have time for. No algebra-capable calculators allowed.

Useful equations: $\frac{\omega}{2\pi} = f = \frac{1}{T}$, $\qquad Q = \frac{m\omega_0}{\gamma}$

1. [9 points]

   (a) Find the general solution to $y'' + y' - 6y = 4e^t + 5e^{2t}$.

(b) If $y(2) = 1$, what is the largest interval in which this gives a unique solution? (don't find this solution)

2. [7 points]

(a) Find the general solution(s) to $(3x + 2y)y' + 3y + 1 = 0$

(b) Find an implicit relation between y and x if the initial conditions are $y(1) = 1$.

(c) BONUS: Find the domain of validity of this solution.

3. [12 points]

You want to design shock absorbers for a mountain bike. Your model is a single damped mass-spring system moving in the vertical direction. The mass is 100 kg (large person plus most of the bike); the spring constant is 10000 kg/s^2.

(a) Find the natural frequency of undamped oscillatory (bouncing) motion, in *cycles per second*.

(b) Find the damping coefficient that is needed to make the system *critically damped*

(c) Write the general solution for the motion of this system in this critically damped case.

(d) An engineer chooses 100 kg/s. Find the Q-factor. Given that bumps on the road drive the system at all frequencies, do you think this is a good choice for a shock absorber? (Why or why not)

(e) With 100 kg/s choice, write the general solution to the motion (using numbers inserted rather than symbols), and sketch the graph of a typical motion after excitation by some initial conditions (*e.g.* bump in the road). Show the envelope and give its decay rate.

(f) If an external driving of the form $F(t) = e^{-10t}$ were applied to the bike system in the critically-damped case above, state and justify the form of particular solution you would use to solve for the motion. (don't solve unless bored).

4. [9 points]

(a) Find the general solution to $(x^2+2)y''+4xy'+2y=0$ using power series (about $x_0=0$). Please write your answer in the form $c_1y_1(x)+c_2y_2(x)$. You must give the first three nonzero terms in the series for both y_1 and y_2. [Do not attempt to write the general term].

(b) What is the best lower bound for the radius of convergence of the power series in your solution? (Explain)

5. [8 points] The equation

$$t^2y'' - 2y = g(t)$$

has the general solution $y(t) = c_1t^2 + c_2/t$ in the case when $g(t) = 0$ for all t.

(a) Find a *particular solution* in the case of general $g(t)$ [your formula will involve $g(t)$.]

(b) Using this or otherwise, find the *general* solution when $g(t) = 1/t$.

(c) What is the largest domain in which the solution must be unique given initial conditions $y(-2) = 2$?

6. [14 points] Slightly shorter questions.

(a) Find a general solution to $dy/dx + y^2 \sin x = 0$.

(b) Is the solution to $y' = -\sqrt{1-y^2}$ with $y(0) = 1$ guaranteed to be unique in some small time interval around zero? (explain what tests you did)

[BONUS: illustrate this by finding the solution(s)]

(c) Consider $(\ln t)y'' + t^{-1}y' + (\sin t)y = 0$. State as much as you can about the Wronskian (as a function of t) of two linearly-independent solutions [Hint: don't solve].

(d) Assume $y_1(t)$ and $y_2(t)$ are solutions of the equation

$$y'' + \cos(t)y' + \tan(t)y = 0$$

and that the Wronskian $W(y_1, y_2)$ takes the value 3 at $t = 0$. On what interval can you be sure that $W(y_1, y_2)(t) \neq 0$?

(e) State how the long-time stability of solutions to $y' = ry + c$ depends on the constant r.