MATH 23 EXAM 1 REVIEW PROBLEMS

Problem 1. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Problem 2. For large, rapidly falling objects, the drag force is approximately proportional to the square of the velocity.

(a) Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.
(b) Determine the limiting velocity after a long time.
(c) If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.
(d) Using the data in part (c), draw a direction field for the differential equation.

Problem 3. According to Newton's law of cooling, the temperature $u(t)$ of an object satisfies the differential equation

$$\frac{du}{dt} = -k(u - T)$$

where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.

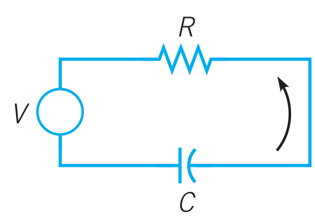
(a) Find the temperature of the object at any time t.
(b) Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

Problem 4. Consider an electric circuit containing a capacitor, resistor, and battery; see the figure below. The charge $Q(t)$ on the capacitor satisfies the equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V\,,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

(a) If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus r.
(b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.



Problem 5. Find the solution to the initial value problem

$$y' + 2y = te^{-2t}, \qquad y(1) = 0\,.$$

Problem 6. Consider the initial value problem

$$y' + \frac{1}{2} = 2\cos(t), \qquad y(0) = -1\,.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

Problem 7. Consider the initial value problem

$$y' - \frac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0\,.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

Problem 8. Consider the initial value problem

$$y' = \frac{x(x^2+1)}{4y^3}, \qquad y(0) = -\frac{1}{\sqrt{2}}\,.$$

(a) Solve the IVP.
(b) Determine the interval on which the solution is valid.

Problem 9. Consider the initial value problem

$$y' = \frac{ty(4-y)}{1+t}, \qquad y(0) = y_0 > 0\,.$$

(a) Determine how the solution behaves as $t \to \infty$.
(b) If $y_0 = 2$, find the approximate time T at which the solution first reaches the value 3.99.

Problem 10. Using a theorem proved in class, determine an interval on which the solution of the following initial value problem is guaranteed to exist. Be sure to state how you are using the theorem.

$$(\ln(t))y' + y = \cot(t), \qquad y(2) = 3$$

Problem 11. State where in the t, y-plane the hypotheses of Theorem 2.4.2 are satisfied for the following ODE.

$$\frac{dy}{dt} = \frac{(\cot(t))y}{1+y}\,.$$

Problem 12. Heat transfer from a body to its surrounds by radiation, based on the Stefan-Boltzmann law, is described by the differential equation

$$\frac{du}{dt} = -\alpha(u^4 - T^4)\,,$$

where $u(t)$ is the absolute temperature of the body at time t, T is the absolute temperature of the surroundings, and α is a constant. However if u is much larger than T, the solutions to the above equation are well approximated by solutions of the simpler equation

$$\frac{du}{dt} = -\alpha u^4\,. \tag{1}$$

Suppose that a body with initial temperature 2000 K is surrounded by a medium with temperature 300 K and that $\alpha = 2 \cdot 10^{-12} \dfrac{1}{\mathrm{K}^3\mathrm{s}}$.

(a) Determine the temperature of the body at any time t by solving (1).
(b) Describe the solution's behavior as $t \to \infty$. Does this behavior make sense?

Problem 13. In this problem, we will attempt to find the curve along which a particle will slide without friction in the minimum time from a given point P to another point Q that is lower than P.

It is convenient to take the upper point P as the origin, and to orient the axes as shown below. The lower point Q has coordinates (x_0, y_0). One can show that the desired curve is given by a function $y(t)$ satisfying the differential equation

$$(1 + (y')^2)y = k^2$$

where k^2 is some positive constant.

(a) Solve the above equation for y'. Why is it necessary to choose the positive square root?
(b) Introduce the new variable t given by the equation

$$y = k^2 \sin^2(t)\,.$$

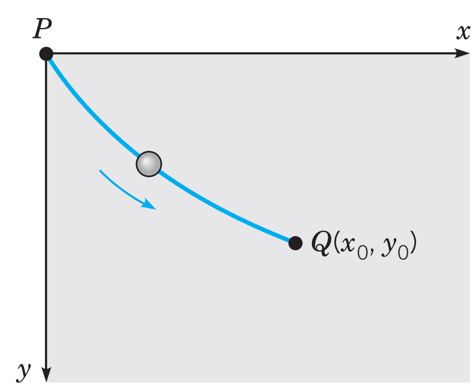
Show that the equation found in part (a) then takes the form

$$2k^2 \sin^2(t)\, dt = dx\,. \qquad (2)$$

(c) Letting $\theta = 2t$, show that the solution of (2) for which $x = 0$ when $y = 0$ is given by

$$x(\theta) = \frac{k^2(\theta - \sin\theta)}{2}, \qquad y(\theta) = \frac{k^2(1 - \cos\theta)}{2}\,.$$

(The graph of the equations $(x(\theta), y(\theta))$ is called a *cycloid*.)
(d) If we make a proper choice of the constant k, then the cycloid also passes through the point (x_0, y_0) and is the solution of to the problem described at the beginning of the exercise. Find k if $x_0 = 1$ and $y_0 = 2$.



Problem 14. Consider the autonomous ODE below. Let $f(y) = dy/dt$.

$$\frac{dy}{dt} = y^2(4 - y^2) \qquad -\infty < y_0 < \infty$$

(a) Sketch the graph of $f(y)$ vs. y by hand (i.e., without using a graphing utility, such as a graphing calculator).
(b) Determine the equilibria, and classify each one as stable, unstable, or semistable.
(c) Draw the phase line, and sketch several graphs of solutions in the t, y-plane.

Problem 15. Consider the autonomous ODE below. Let $f(y) = dy/dt$.

$$\frac{dy}{dt} = y^2(1-y)^2 \qquad -\infty < y_0 < \infty$$

(a) Sketch the graph of $f(y)$ vs. y by hand (i.e., without using a graphing utility, such as a graphing calculator).
(b) Determine the equilibria, and classify each one as stable, unstable, or semistable.
(c) Draw the phase line, and sketch several graphs of solutions in the t, y-plane.

Problem 16. Determine whether the following equation is exact. If it is exact, find the solution.

$$(2x+3)\,dx + (2y-2)\,dy = 0\,.$$

Problem 17. Find an integrating factor and solve the following equation.

$$y + (2x - ye^y)y' = 0\,.$$

Problem 18. Solve the following equation.

$$(3x^2y + 2xy + y^3)\,dx + (x^2 + y^2)\,dy = 0\,.$$

Problem 19. Find the general solution of the following differential equation.

$$y'' + 2y' - 3y = 0$$

Problem 20.

(a) Solve the following initial value problem.

$$6y'' - 5y' + y = 0, \qquad y(0) = 4, \qquad y'(0) = 0$$

(b) Describe the solution's behavior as $t \to \infty$.

Problem 21.

(a) Find the solution to the initial value problem

$$2y'' - 3y' + y = 0, \qquad y(0) = 2 \qquad y'(0) = 1/2\,.$$

(b) Determine the maximum value of the solution.

Problem 22. Find the Wronskian of the given pairs of functions.

(a) $e^t \sin(t), \quad e^t \cos(t)$
(b) $\cos^2(\theta), \quad 1 + \cos(2\theta)$

Problem 23.

(a) Verify that $y_1(t) = 1$ and $y_2(t) = \sqrt{t}$ are solutions to the differential equation

$$yy'' + (y')^2 = 0$$

for $t > 0$.

(b) Show that $y = c_1 + c_2\sqrt{t}$ is not, in general, a solution to the above equation.

(c) Explain why this does not contradict Theorem 3.2.2.

Problem 24.

(a) Consider the differential equation

$$y'' + 4y = 0\,.$$

Verify that the functions $y_1 = \cos(2t)$ and $y_2 = \sin(2t)$ are solutions. Do they form a fundamental set of solutions?

(b) Determine a fundamental set of solutions to the following differential equation.

$$y'' + 4y' + 3y = 0$$

Problem 25. For each of the differential equations below, find the general solution.

(a) $y'' + 2y' + 2y = 0$

(b) $y'' + 2y' - 8y = 0$

Problem 26. For each of the initial value problems below, find a solution and describe its behavior as $t \to \infty$.

(a) $y'' + 4y = 0$, $\quad y(0) = 0$, $\quad y'(0) = 1$

(b) $y'' + y' + (5/4)y = 0$, $\quad y(0) = 3$, $\quad y'(0) = 1$