MATH 23 Exam 2 Review Problems

Problem 1. Use the method of reduction of order to find a second solution of the given differential equation

\[ x^2 y'' - (x - 0.1875)y = 0, \quad x > 0, \quad y_1(x) = x^{1/4}e^{2\sqrt{x}} \]

Problem 2. Consider the initial value problem

\[ 4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2. \]

(a) Solve the initial value problem and plot the solution.

(b) Determine the coordinates \((t_M, y_M)\) of the maximum point.

(c) Change the second initial condition to \(y'(0) = b > 0\) and find the solution as a function of \(b\).

(d) Find the coordinates \((t_M, y_M)\) of the maximum point in terms of \(b\). Describe the dependence of \(t_M\) and \(y_M\) on \(b\) as \(b\) increases.

Problem 3. Solve the given initial value problem. Sketch the graph of the solution and describe its behaviour for increasing \(t\).

\[ 9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2 \]

Problem 4. Find the general solution of the given differential equation

\[ u'' + \omega_0^2 u = \cos \omega t, \quad \omega^2 \neq \omega_0^2 \]

Problem 5. Determine a suitable form of particular solution \(Y(t)\) using the method of undetermined coefficients

\[ y'' + 3y' + 2y = e^t(t^2 + 1) \sin 2t + 3e^{-t} \cos t + 4e^t \]
Problem 6. Verify that the given functions \( y_1 \) and \( y_2 \) satisfy the corresponding homogenous equation; then find a particular solution of the given nonhomogeneous equation.

\[
x^2y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0, \quad y_1(x) = x^2, \quad y_2(x) = x^2 \ln x
\]

Problem 7. Verify that the given functions \( y_1 \) and \( y_2 \) satisfy the corresponding homogenous equation; then find a particular solution of the given nonhomogeneous equation.

\[
x^2y'' + xy' + (x^2 - 0.25)y = g(x), \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x, \quad y_2(x) = x^{-1/2} \cos x
\]

Problem 8. A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position \( u \) of the mass at any time \( t \). Determine the frequency, period, amplitude, and phase of the motion.

Problem 9. A 1/4-kg mass is attached to a spring with a stiffness 4 N/m. The damping constant \( b \) for the system is 1 N-sec/m. If the mass is displaced 1/2 m to the left and an initial velocity of 1 m/sec to the left, find the equation of motion. What is the maximum displacement that the mass will attain?

Problem 10. Verify that the given vector satisfies the given differential equation

\[
x' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} x, \quad x = \begin{bmatrix} 6 \\ -8 \\ -4 \end{bmatrix} e^{-t} + 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-2t}
\]

Problem 11. Verify that the given matrix satisfies the given differential equation

\[
\psi' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \psi, \quad \psi(t) = \begin{bmatrix} e^t & e^{-2t} & e^{3t} \\ -4e^t & -e^{-2t} & 2e^{3t} \\ -e^t & -e^{-2t} & e^{3t} \end{bmatrix}
\]

Problem 12. Verify that the given functions are solutions of the differential equation, and determine the Wronskian

\[
xy''' - y'' = 0; \quad 1, \quad x, \quad x^3
\]
Problem 13. Determine intervals in which solutions are sure to exist
\[(x - 1)y^{(4)} + (x + 1)y'' + (\tan x)y = 0\]

Problem 14. Find the general solution of the given differential equation
\[y^{(6)} - y'' = 0\]

Problem 15. Find the general solution of the given differential equation
\[y^{(8)} + 8y^{(4)} + 16y = 0\]

Problem 16. Determine a suitable form for the particular solution \(Y(t)\) if the method of undetermined coefficients is to be used. Do not evaluate the constants.
\[y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t\]

Problem 17. Find the solution of the initial value problem
\[y''' - 3y'' + 2y' = t + e^t; \quad y(0) = 1, \quad y'(0) = -1/4, \quad y''(0) = -3/2\]

Problem 18. Given that \(x, x^2, \text{ and } 1/x\) are solutions of the homogeneous equation corresponding to
\[x^3y''' + x^2y'' - 2xy' + 2y = 2x^4, \quad x > 0,\]
determine a particular solution.

Problem 19. Find the solution of the given initial value problem
\[y''' + y' = \sec t; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -2\]

Problem 20. Find all eigenvalues and eigenvectors of the given matrix
\[
\begin{bmatrix}
1 & i \\
-i & 1
\end{bmatrix}
\]

Problem 21. Find all eigenvalues and eigenvectors of the given matrix
\[
\begin{bmatrix}
11/9 & -2/9 & 8/9 \\
-2/9 & 2/9 & 10/9 \\
8/9 & 10/9 & 5/9
\end{bmatrix}
\]