Math 23: Differential Equations
Final Exam

November 16, 2018

NAME: ____

SECTION (check one box): Section 1 (A. Gelb 9L) ☐
Section 1 (S. Lin 2) ☐

Instructions:

1. Please turn off or silence cell phones and other electronic devices which may be disruptive.
2. Unless otherwise stated, you must justify your solutions to receive full credit. We will not grade illegible work, or work that is scratched out.
3. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
4. There is a table of Laplace transforms, trigonometric identities and some useful integral formulas on the back page.
5. This exam is **closed book**. You may not use notes, or other external resource. You may use calculators. It is a violation of the honor code to give or receive help on this exam. You may of course ask for clarification from the exam proctor on any problem.

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: ____

Problem	Points	Score
1	35	
2	20	
3	40	
4	20	
5	25	
6	30	
7	30	
Total	200	

1. *[35 points]* TRUE/FALSE: You must provide a concise justification for your answer. If you claim the statement is false, a counter-example is sufficient.

   (a) For all values $a > 0$, solutions to $y'' + 10y' + ay = 0$ tend to zero.

   (b) The system of equations $x' = 2x - 3x^2 - 2xy; y' = y - .5y^2 - 3xy$ describes a competitive species model where the two species x and y can peacefully coexist.

   (c) Let $\mathbf{x}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}$. If $b > 0$ and $c > 0$ then the critical point is not a spiral.

   (d) $f * 1 = 1 * f = f$, where $*$ is the convolution operator.

(e) Let $y_1 = e^t$ and $y_2 = e^{t-1}$. All solutions of the equation $y'' - y = 0$ can be written as $y = c_1 y_1 + c_2 y_2$ where c_1 and c_2 are any constants.

(f) The critical point $(4, 3)$ of the system given below is an asymptotically stable spiral point.

$$\begin{aligned} \frac{dx}{dt} &= 33 - 10x - 3y + x^2 \\ \frac{dy}{dt} &= -18 + 6x + 2y - xy \end{aligned}$$

(g) There is a unique solution to $y' = \frac{1}{2}(-t + \sqrt{t^2 + 4y})$, $t > 0$, with initial conditions $y(2) = -1$ in some interval $2 - h < t < 2 + h$ contained in $0 < t < \infty$ and $-\infty < y < \infty$.

2. [20-points] Find the inverse Laplace transforms of the following functions. For your convenience there is a Laplace transform table, some trigonometric identities, and some integral formulas on the back page of your exam.

   (a) $\frac{1}{(s^2+1)(s^2+4)}$

   (b) $\frac{s}{(s^2+1)(s^2+4)}$

3. [40-points] Consider the function

$$f(t) = \begin{cases} 0, & 0 \le t \le \frac{\pi}{2}, \\ \\ \sin t, & \frac{\pi}{2} \le t < \pi, \\ \\ 0, & t \ge \pi. \end{cases}$$

(a) Sketch $f(t)$.

(b) Rewrite $f(t)$ in terms of unit step functions.

(c) Compute $\mathcal{L}\{f(t)\}$.

(d) Use the results in (c) to solve the initial value problem

$$y'' + 4y = f(t), \qquad y(0) = 0, y'(0) = 0.$$

Hint: Use your solutions from problem 2.

4. *[20- points]* Solve the initial value problem

$$y'' + 4y' + 5y = \delta(t - \pi), \qquad y(0) = 0, y'(0) = 2.$$

5. *[25- points]* Consider the model:

$$\frac{dy}{dt} = f(y) = (y^2 - 1)(y - 2). \tag{1}$$

We wish to analyze the behavior of y for $t > 0$.

(a) Graph $f(y)$ versus y, and explain what can be learned from your graph about the behavior of y.

(b) Determine all equilibrium solutions.

(c) Draw the corresponding **phase line** for eq. (1).

(d) Determine the stability of each equilibrium point. Explain your results.

6. *[30- points]* Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{2}$$

(a) Find matrices $\mathbf{T}$ and $\mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix, such that $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$.

(b) Using $\mathbf{T}$ and $\mathbf{D}$ from (2), let $\mathbf{x} = \mathbf{T}\mathbf{y}$ and construct the system

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}(t), \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{3}$$

Here $\mathbf{h}(t) = \mathbf{T}^{-1}\mathbf{g}(t)$, and $\mathbf{g}(t) = \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$.

(c) Solve (3) to find the general solution for $\mathbf{y}$.

(d) Use your solution for y to find the general solution for $\mathbf{x}$ in (2).

7. *[30- points]* Consider the following predator-prey system where x is the prey and y is the predator:

$$\begin{aligned} x' &= 200x - 4xy \\ y' &= -150y + 2xy \end{aligned}$$

(a) Determine the system's critical points.

(b) Write the corresponding linearized system corresponding to each critical point.

(c) Describe the type and the stability of each critical point for the corresponding linearized systems. Be as specific as possible.

(d) Sketch a phase plane portrait for the predator-prey system.

Trigonometric identities:

1. $\sin^2 x + \cos^2 x = 1$.
2. $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$.
3. $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$.
4. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.
5. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

Integrals:

1. $\int \sin ax \sin bx dx = \frac{1}{2}\left(\frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}\right) + C$ for $a \neq b$
2. $\int \cos ax \cos bx dx = \frac{1}{2}\left(\frac{\sin(a-b)x}{a-b} + \frac{\sin(a+b)x}{a+b}\right) + C$ for $a \neq b$
3. $\int \sin ax \cos bx dx = \frac{1}{2}\left(\frac{\cos(a-b)x}{a-b} - \frac{\cos(a+b)x}{a+b}\right) + C$ for $a \neq b$
4. $\int \sin^2 ax dx = \frac{1}{2}\left(x - \frac{\sin 2ax}{2a}\right) + C$
5. $\int \cos^2 ax dx = \frac{2ax + sin 2ax}{4a} + C$
6. $\int \cos ax \sin ax dx = -\frac{\cos^2 ax}{2a} + C$

Table 1: Laplace Transform Table.

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$, $s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s > \|a\|$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$, $s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}