Math 23, Final
June 1 2017

Name ______________________________ (please print)

Instructor: *Edgar Costa* or *Anne Gelb* (please circle one)

Instructions

- Please **print your name** in the blank space above.
- Please **circle your instructor**.
- Please **turn off cell phones** or other electronic devices which may be disruptive.
- Present your work neatly and clearly. **Justify your answers completely**. Unless explicitly told otherwise, you will not receive full credit for insufficiently justified answers. Please box your answers, when appropriate.
- It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
- This exam is **closed book**. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource. It is a violation of the honor code to give or receive help on this exam. However, you may ask the instructor for clarification on problems.
- Consider signing the FERPA waiver:

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructors office to retrieve my examination paper. FERPA waiver signature:

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: ______________________________

Grader's use only:

1. ___________ /40

2. ___________ /40

3. ___________ /15

4. ___________ /15

5. ___________ /20

6. ___________ /20

7. ___________ /10

8. ___________ /20

9. ___________ /20

Total: ___________ /200

1. *[40 points]* TRUE or FALSE? Please keep your justifications short. In some cases, drawing a figure might aid your justification.

   (a) $c_1 + c_2\sqrt{t}$ is a general solution to the differential equation $yy'' + (y')^2 = 0$ for $t > 0$.

   (b) The solutions $y_1(x) = x^2$ and $y_2(x) = \frac{1}{x^2}$ form a fundamental set of solutions for the problem $x^2y'' + xy' - 4y = 0$ on $(-\infty, \infty)$.

   (c) The initial value problem $y' = 2\sqrt{y}$, $y(0) = 0$ has a unique solution.

   (d) Consider the differential equation $X' = A \cdot X$, where A is a $n \times n$ matrix. If all eigenvalues of A have their real part less than zero, then $0 \in \mathbb{R}^n$ is asymptotically stable.

   (e) If y_1 and y_2 are two solutions of an nonhomogeneous equation $ay'' + by' + cy = f(x)$, then their sum is a solution of the equation $ay'' + by' + cy = 0$

   (f) Consider the differential equation $X' = A \cdot X$, where A is a $n \times n$ matrix. If all eigenvalues of A have their real part less than or equal to zero, then $0 \in \mathbb{R}^n$ is stable.

   (g) There is a solution to the ODE $y'' - 4y' + 4y = \cos(2t)$ of the form $y_p(t) = At\cos(2t) + Bt\sin(2t)$ where A and B are nonzero constants.

   (h) Consider the differential equation $X' = A \cdot X$, where A is a $n \times n$ matrix. The only equilibrium solution is $X(t) = 0 \in \mathbb{R}^n$.

2. *[40 points]* Consider the system:

$$\begin{cases} x' = x + \alpha y \\ y' = x + y \end{cases}$$

(a) Show that the critical point $(0, 0)$ is a nodal source for all $0 < \alpha < 1$.

(b) Write the general solution for $0 < \alpha < 1$ and sketch the corresponding phase portrait.

(c) Classify the critical point $(0, 0)$ for $\alpha = 0$.

(d) Classify the critical point $(0, 0)$ for $\alpha < 0$.

3. *[15 points]* Solve the initial value problem

$$y'' + 4y' + 5y = \delta(t - \pi),$$

with $y(0) = 0$ and $y'(0) = 2$.

4. *[15 points]* Find the general solution for $y'' + 4y' + 4y = t^{-2}e^{-2t}$.

5. *[20 points]* Consider the initial value problem $y' = -2ty^4, y(0) = y_0$.

(a) Solve the initial value problem.

(b) Determine how the interval in which the solution exists depends on the initial value y_0.

6. *[20 points]* Consider the system $\begin{cases} x' &= 1 + y^4 - 4x^2 \\ y' &= 8xy. \end{cases}$

(a) Find the critical points and determine their stability.

(b) There is a nonconstant solution curve $\{(x(t), y(t))\}_{t\in\mathbb{R}} \subset \mathbb{R}^2$ which is bounded for all $t \in \mathbb{R}$. Describe the curve $\{(x(t), y(t))\}_{t\in\mathbb{R}} \subset \mathbb{R}^2$ in the phase plane.

7. *[10 points]* Sketch the phase portrait of the system

$$
\begin{aligned}
x' &= \frac{2y}{x^2 + y^2 + 1} \\
y' &= -\frac{2x}{x^2 + y^2 + 1}.
\end{aligned}
$$

8. *[20 points]*

(a) Solve the initial value problem

$$\begin{cases} x' &= x - y \\ y' &= 2x - y \end{cases} \qquad x(0) = 1,\ y(0) = 2$$

(b) Solve the initial value problem

$$\begin{cases} x' &= x - y \\ y' &= 2x - y \\ z' &= y - \cos(t)z \end{cases} \qquad x(0) = 1,\ y(0) = 2,\ z(0) = 3.$$

9. *[20 points]* Match the following system of differential equations with the vector fields.

(a) $\begin{cases} x' = y \\ y' = x^3 - x \end{cases}$ ☐

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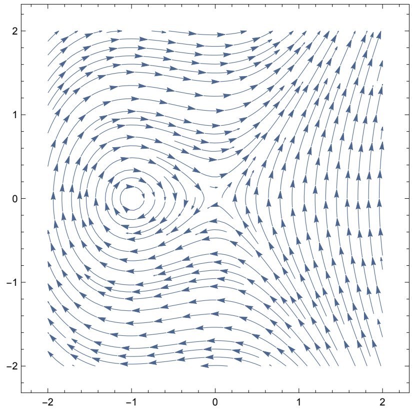

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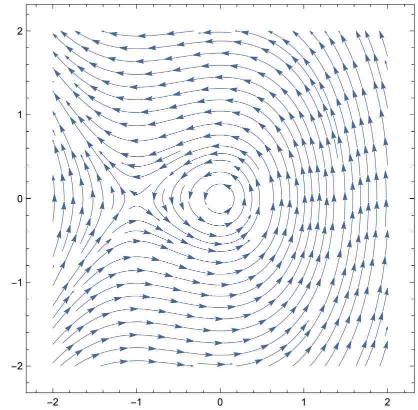

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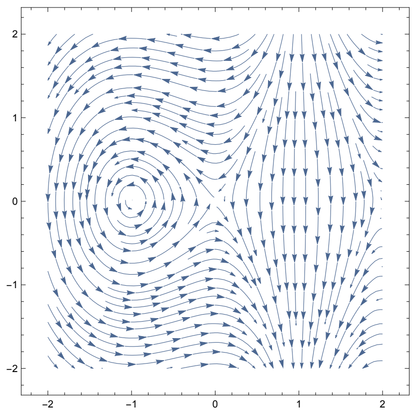

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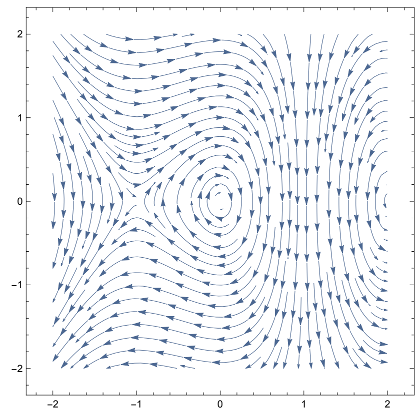

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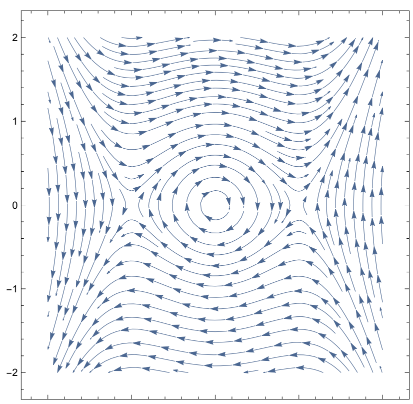

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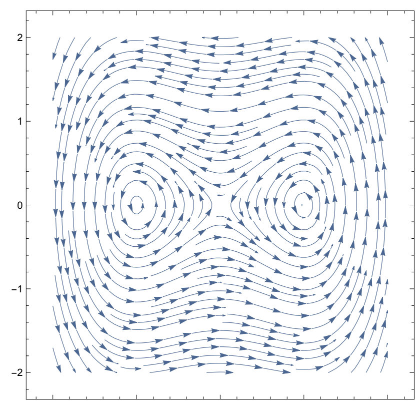
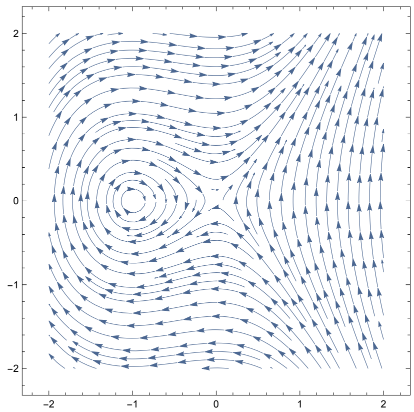
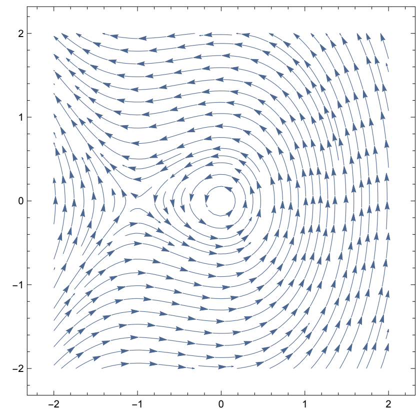
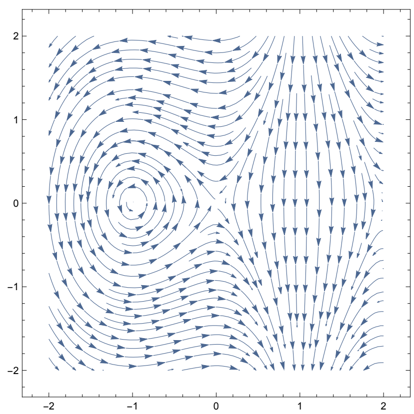
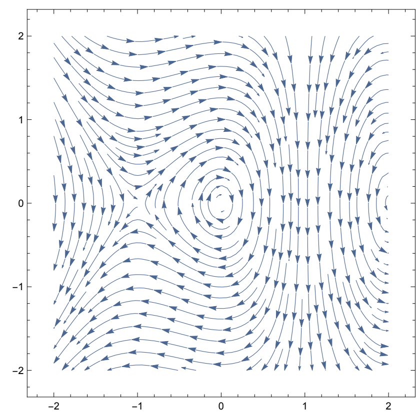


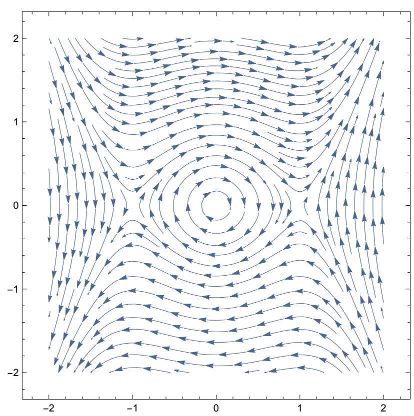
(b) $\begin{cases} x' = y(x-1) \\ y' = x(x+1)(x-2) \end{cases}$ ☐

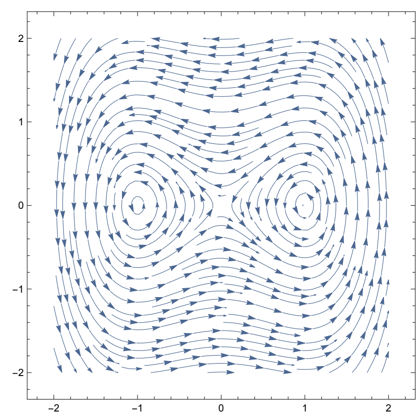
A: 

B: 

C: 

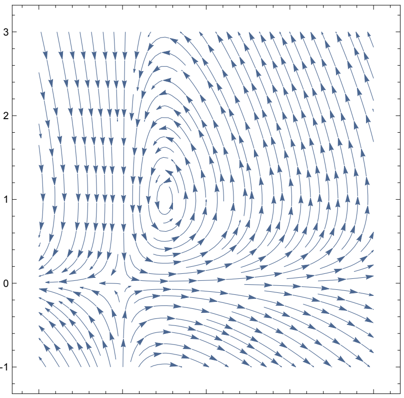
D 

E: 

F 

(c) $\begin{cases} x' = x(6 - 2y - 4x) \\ y' = y(6 - 2x - 3y) \end{cases}$

A:



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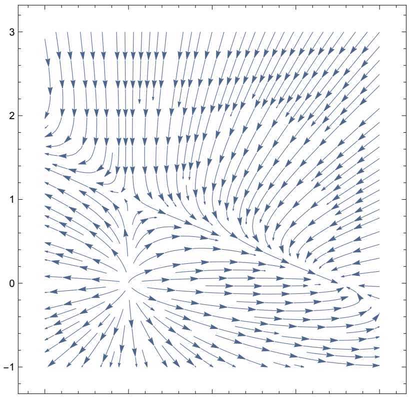

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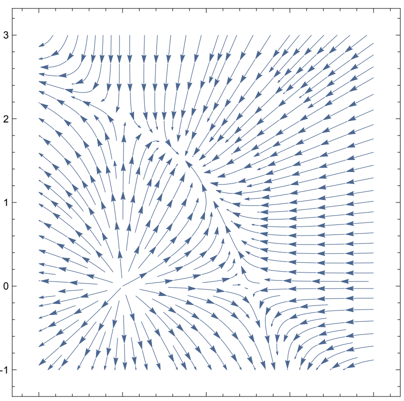

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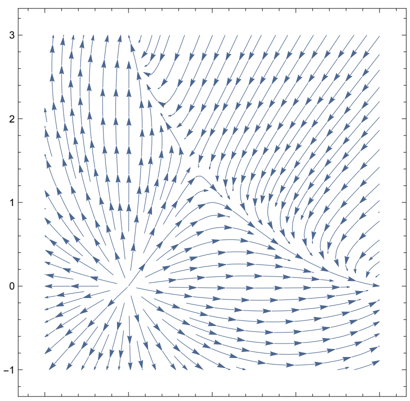

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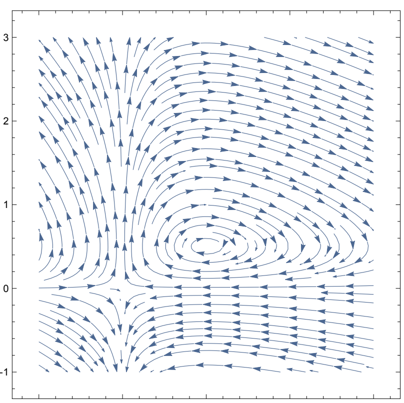

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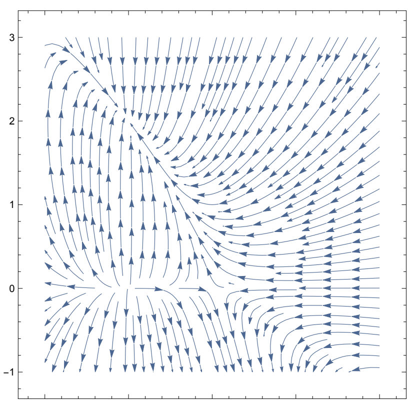

(d) $\begin{cases} x' = x(1-2y) \\ y' = y(x-1) \end{cases}$ $\square$

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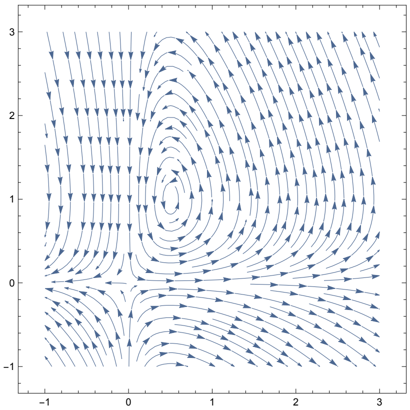

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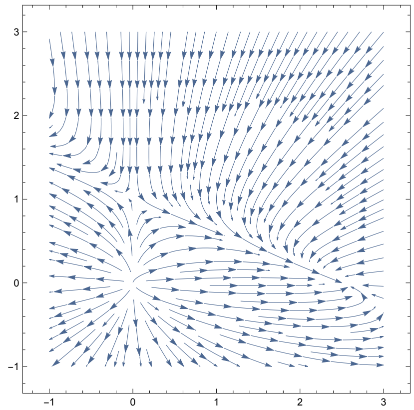

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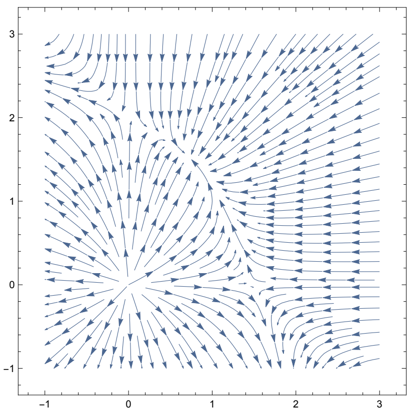

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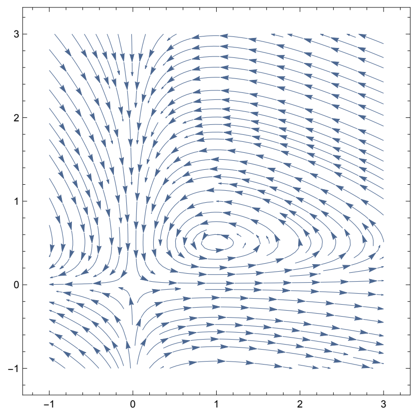

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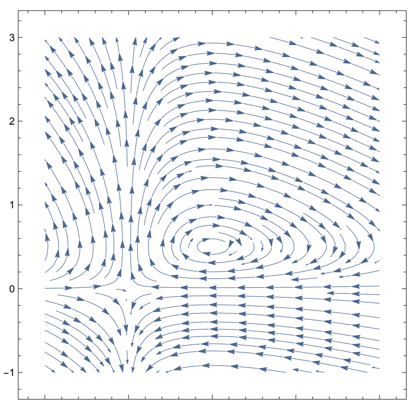

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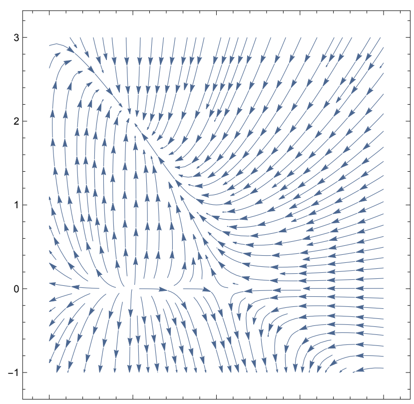

Table 1: Laplace Transform Table.

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$, $s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s > \|a\|$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$, $s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}

Common integral formulas:

1. $\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
2. $\int \cos^2 u du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
3. $\int u \cos u du = \cos u + u \sin u + C$
4. $\int u \sin u du = \sin u - u \cos u + C$
5. $\int e^{au} \sin bu du = \frac{e^{au}}{a^2+b^2}(a \sin bu - b \cos bu) + C$
6. $\int e^{au} \cos bu du = \frac{e^{au}}{a^2+b^2}(a \cos bu + b \sin bu) + C$