MATH 23, §3.5 + 4.3

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Some particular solutions

$$\underbrace{c_n y^{(n)} + \cdots + c_1 y' + c_0 y}_{\text{order } n \text{ with constant coefficients}} = g(t) \qquad (\Delta)$$

Write

$$p(r) := c_n r^n + \cdots + c_1 r + c_0$$

Characteristic equation is given by $p(r) = 0$

$g(t)$ — RHS of (Δ)	$y_p(t)$ — a particular solution for (Δ)
$a_k t^k + \cdots + a_1 t + a_0$	$t^s(A_k t^k + \cdots + A_1 t + A_0)$ s = number of times 0 is a root of $p(r)$
$e^{\alpha t}(a_k t^k + \cdots + a_1 t + a_0)$	$t^s e^{\alpha t}(A_k t^k + \cdots + A_1 t + A_0)$ s = number of times α is a root of $p(r)$
$e^{\alpha t}\cos(\beta t)(a_k t^k + \cdots + a_1 t + a_0)$ $+$ $e^{\alpha t}\sin(\beta t)(b_k t^k + \cdots + b_1 t + b_0)$ we can have a_i or $b_i = 0$	$e^{\alpha t} t^s \cos(\beta t)(A_k t^k + \cdots + A_1 t + A_0)$ $+$ $e^{\alpha t} t^s \sin(\beta t)(B_k t^k + \cdots + B_1 t + B_0)$ s = number of times $\alpha + i\beta$ is a root of $p(r)$

1. Example: $y'' + y' + y = t^3 + 1$

1.1. **Homogeneous solutions.**

- Homogeneous equation: $y'' + y' + y = 0$
- Characteristic equation: $r^2 + r + 1 = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
- General solution for the homogeneous equation:

$$y(t) = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right)$$

1.2. **Finding a particular solution.**

- $g(t) = t^3 + 1$ a polynomial of degree 3
- $s = 0$, as 0 is **not** a root of $r^2 + r + 1$ ($0^2 + 0^1 + 1 \neq 0$)
- $y_p(t)$ is of the shape $A_3t^3 + A_2t^2 + A_1t + A_0$
- $\begin{cases} y_p(t) & = A_3t^3 + A_2t^2 + A_1t + A_0 \\ y_p'(t) & = 3A_3t^2 + 2A_2t + A_1 \\ y_p''(t) & = 6A_3t + 2A_2 \end{cases}$
- $y_p'' + y_p' + y_p = t^3 + 1 \Rightarrow$

$$(6A_3t + 2A_2) + (3A_3t^2 + 2A_2t + A_1) + (A_3t^3 + A_2t^2 + A_1t + A_0) = t^3 + 1$$

$$\Leftrightarrow$$

$$A_3t^3 + (3A_3 + A_2)t^2 + (6A_3 + 2A_2 + A_1)t + (2A_2 + A_1 + A_0) = 1t^3 + 0t^2 + 0t + 1$$

$$\Leftrightarrow \begin{cases} 1 & = A_3 \\ 0 & = 3A_3 + A_2 \\ 0 & = 6A_3 + 2A_2 + A_1 \\ 1 & = 2A_2 + A_1 + A_0 \end{cases} \Leftrightarrow \begin{cases} A_3 & = 1 \\ A_2 & = -3 \\ A_1 & = 0 \\ A_0 & = 7 \end{cases}$$

- A particular solution for

$$y'' + y' + y = t^3 + 1$$

is

$$t^3 - 3t^2 + 7$$

1.3. **Altogether.**

- A general solution for

$$y'' + y' + y = t^3 + 1$$

is

$$y = c_1 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + t^3 - 3t^2 + 7$$