HW 1 Key

1.3

For the following differential equations, determine if they are linear and determine their order.

1) $t^2y'' + ty' + 2y = \sin(t)$
order: 2
linear

2) $(1 + y^2)y'' + ty' + y = e^t$
order: 2
non-linear

3) $y^{(4)} + y''' + y'' + y = 1$
order: 4
linear

4) $y' + ty^2 = 0$
order: 1
non-linear

5) $y'' + \sin(t + y) = \sin(t)$
order: 2
non-linear

6) $y''' + ty' + \cos^2(t)y = t^3$
order: 3
linear

Check that the given equations are solutions to the given differential equation.

11) $2t^2y'' + 3ty' - y = 0$, $t > 0$

1. $y(t) = t^{1/2}$

$$y' = \frac{1}{2}t^{-1/2}$$
$$y'' = -\frac{1}{4}t^{-3/2}$$

Check:

$$2t^2(-\frac{1}{4}t^{-3/2}) + 3t(\frac{1}{2}t^{-1/2}) - t^{1/2} = 0$$

2. $y(t) = t^{-1}$

$$y' = -t^{-2}$$
$$y'' = 2t^{-3}$$

Check:

$$2t^2(2t^{-3}) + 3t(-t^{-2}) - t^{-1} = 0$$

13) $y'' + y = \sec(t)$, $0 < t < \pi/2$

$$y = \cos(t)\ln(\cos(t)) + t\sin(t)$$
$$y' = -\sin(t)\ln(\cos(t)) - \sin(t) + \sin(t) + t\cos(t)$$
$$y'' = -\cos(t)\ln(\cos(t)) + \sin^2(t)/\cos(t) + \cos(t) - t\sin(t)$$

Check:

$$\sin^2(t)/\cos(t) + \cos(t) = (\sin^2(t) + \cos^2(t))/\cos(t) = \sec(t)$$

15) Find r such that $y = e^{rt}$ is a solution to the differential equation

$$y' + 2y = 0.$$

If $y = e^{rt}$, then $y' = re^{rt}$. Plugging in to the differential equation, we get: $re^{rt} + 2e^{rt} = 0$. Since $e^{rt} > 0$ for all r and t, we can divide both sides by e^{rt} and see that the equation is solved exactly when $r + 2 = 0$ and thus $r = -2$.

2.1

Find explicit solutions to the following initial value problems.

13) $y' - y = 2te^{2t}$, $y(0) = 1$
Integrating factor: $\mu(t) = \exp \int(-1)\,dt = e^{-t}$.

$$e^{-t}y = \int 2te^{2t}\,dt$$

and use integration by parts to get

$$e^{-t}y = 2(te^t - e^t) + C$$

use initial condition

$$1 = 0 - 2 + C; \quad C = 3$$

Therefore, we obtain $y = 2e^{2t}(t-1) + 3e^t$ as the solution.

16) $y' + (2/t)y = \cos(t)/t^2$, $y(\pi) = 0$, $t > 0$
Integrating factor: $\exp \int 2/t\,dt = e^{2\ln|t|} = t^2$.

$$t^2 y = \int \cos(t)\,dt = \sin(t) + C$$

$$y = \sin(t)/t^2 + C/t^2$$

use initial condition

$$0 = 0 + C/\pi^2; \quad C = 0$$

Therefore, we obtain $y = \sin(t)/t^2$ as the solution.

24) $ty' + (t+1)y = 2te^{-t}$, $y(1) = a$, $t > 0$
(a) $y \to 0$ as $t \to \infty$. Some values of a give a solution which tends towards ∞ as t goes to 0, others tend toward $-\infty$.
(b) Integrating factor: $\mu(t) = \exp \int (t+1)/t\, dt = te^t$

$$te^t y = \int 2t\, dt = t^2 + C$$

$$y = \frac{t^2 + C}{te^t}$$

Use initial condition

$$a = \frac{1+C}{e}; \quad C = ae - 1$$

Therefore the solution to the initial value problem is

$$y = te^{-t} + (ae-1)/(te^t).$$

As $t \to 0$,

$$y \to \lim_{t\to 0}[te^{-t} + (ae-1)/(te^t)] = \lim_{t\to 0}[(ae-1)/(te^t)].$$

Therefore, the change in behavior happens when $ae-1 = 0$. So, $a_0 = 1/e$.
(c) Using the critical value, $y = t/e^t$, and we have $y \to 0$ as $t \to 0$.

35) Give a first order linear differential equation, all of whose solutions tend toward $3 - t$ as $t \to \infty$.
We know the equation has the form $y' + p(t)y = g(t)$, so we must determine which $p(t)$ and $g(t)$ give the behavior we want. For simplicity, let's see if we can find a solution where $p(t) = 1$. Let us also assume that one solution will be exactly $y = 3 - t$. Now we may determine $g(t)$.

$$y' + y = g(t)$$

$$-1 + (3-t) = g(t)$$

$$g(t) = 2 - t$$

Now check that all solutions of $y' + y = 2 - t$ do indeed have the desired property.

2.2

Solve the following separable differential equations.

1)

$$dy/dx = x^2/y$$

$$\int y\,dy = \int x^2\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

We leave this as an implicit solution to the differential equation. This solution is valid when $y \neq 0$.

2)

$$dy/dx = \frac{x^2}{y(1+x^3)}$$

$$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{3}\ln|1+x^3| + C$$

We leave this as an implicit solution to the differential equation.This solution is valid when $x \neq -1$ and $y \neq 0$.

3)

$$dy/dx + y^2\sin(x) = 0$$

$$\int \frac{1}{y^2}\,dy = \int -\sin(x)\,dx$$

$$-\frac{1}{y} = \cos(x) + C$$

$$y = \frac{1}{C - \cos(x)}$$

This solution is valid when $y \neq 0$. Also, $y = 0$ is a (constant) solution to the differential equation.

6)

$$x\frac{dy}{dx} = (1-y^2)^{1/2}$$

$$\int (1-y^2)^{-1/2}\,dy = \int \frac{1}{x}\,dx$$

$$\arcsin(y) = \ln|x| + C$$

$$y = \sin(\ln|x| + C)$$

This solution is valid when $y \neq 0$ and $y < 1$. Also $y = 1$ and $y = -1$ are (constant) solutions to the differential equation.

15) (a) $dy/dx = \frac{2x}{1+2y}$, $y(2) = 0$

$$\int 1 + 2y\,dy = \int 2x\,dx$$

$$y + y^2 = x^2 + C$$

Find the value of C using initial conditions.

$$0 = 4 + C$$

$$C = -4$$

We obtain the formula

$$y + y^2 - (x^2 + 4) = 0$$

Using the quadratic formula and the initial condition (to determine $+/-$), we find:

$$y = -\frac{1}{2} + \frac{1}{2}\sqrt{4x^2 - 15}.$$

(c) Certainly, we must have that $4x^2 - 15 \geq 0$, which implies that $|x| \geq \sqrt{15/4}$. Since the initial condition places x on the positive side of this, we must have that $x \geq \sqrt{15/4}$. Finally, since $1 + 2y \neq 0$ (since you can't divide by 0), $y \neq -1/2$. Therefore, $\sqrt{4x^2 - 15} \neq 0$ and thus, $x \neq \sqrt{15/4}$. Therefore, this solution is defined when $x > \sqrt{15/4}$

30) $\frac{dy}{dx} = \frac{y-4x}{x-y}$

(a) Multiply by $1 = \frac{1/x}{1/x}$
(b) Use chain rule: $y = xv(x) \implies dy/dx = v + x\frac{dv}{dx}$.
(c) The equation separates as

$$\frac{1-v}{v^2-4}dv = \frac{1}{x}dx.$$

(d) To handle the left integral above, use partial fraction decomposition. You should get something equivalent to the following:

$$-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| + C.$$

Getting rid of ln, and letting $A = e^C$, we obtain:

$$(v-2)^{-1/4}(v+2)^{-3/4} = Ax$$

(e)

$$(y/x-2)^{-1/4}(y/x+2)^{-3/4} = Ax$$