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The Laplace Transform

$$\left\{ \begin{array}{rl} f : [0, +\infty) & \longrightarrow \mathbb{R} \\ t & \longmapsto f(t) \end{array} \right\} \longmapsto \left\{ \begin{array}{rl} \mathcal{L}(f) : I & \longrightarrow \mathbb{R} \\ s & \longmapsto \mathcal{L}(f)(s) \end{array} \right\}$$

IVP in t-domain $\longmapsto$ algebraic equations in the s-domain

Definition

$$\mathcal{L}(f)(s) := \int_0^{+\infty} e^{-st} f(t)\, \mathrm{d}t \quad \text{(if the integral converges)}$$

Note: $\mathcal{L}$ is a linear operator!In other words, if $\mathcal{L}(f_1)(s)$ and $\mathcal{L}(f_2)(s)$ exist, then

$$\mathcal{L}(c_1 f_1 + c_2 f_2)(s) = c_1 \mathcal{L}(f_1)(s) + c_2 \mathcal{L}(f_2)(s)$$

Some examples

- $\mathcal{L}(1) = \int_0^{+\infty} e^{-st}\, \mathrm{d}t = -\lim_{A \to +\infty} \left[\frac{e^{-st}}{s}\right]_0^A = \frac{1}{s}, \quad s > 0$
- $\mathcal{L}(e^{at}) = \int_0^{+\infty} e^{at} e^{-st}\, \mathrm{d}t = \int_0^{+\infty} e^{(a-s)t}\, \mathrm{d}t = \frac{1}{s-a}, \quad s > a$

  In particular, $\mathcal{L}(e^{0t}) = \frac{1}{s}, \quad s > 0$

Theorem 6.1.2

1. If f is piecewise continuous on $[0, A]$, for any $A > 0$
2. If $|f(t)| \leq Ke^{at}$ for $t > M$, with $K, M, a \in \mathbb{R}$ and $K, M > 0$.

Then the Laplace transform $\mathcal{L}(f)(s)$ exists for $s > a$.

More examples

- $\mathcal{L}(\cos(\beta t)) = ?$
  $\mathcal{L}(\sin(\beta t)) = ?$
  We could use the definition, but what would require integrating by parts twice per function!
- Let's use complex analysis!

$$e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$\left|e^{(\alpha+i\beta)t}\right| = e^{\alpha t}\sqrt{\cos(\beta t)^2 + \sin(\beta t)^2} = e^{\alpha t}$$

$$\mathcal{L}\left(e^{(\alpha+i\beta)t}\right) = \frac{1}{s-(\alpha+\beta i)}, \quad s > \alpha$$

Exercise

Deduce $\mathcal{L}\left(e^{\alpha t}\cos(\beta t)\right)$ and $\mathcal{L}\left(e^{\alpha t}\sin(\beta t)\right)$ with $\alpha, \beta \in \mathbb{R}$.

Main Theorem

Theorem 6.2.1

1. If f is continuous and f' is piecewise continuous on $[0, A]$, for any $A > 0$
2. If $|f(t)| \leq Ke^{at}$ for $t > M$, with $K, M, a \in \mathbb{R}$ and $K, M > 0$.

Then the Laplace transform $\mathcal{L}(f')(s)$ exists for $s > a$ and

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$$

Proof sketch: If f and f' are *continuous* on $[0, A]$, then

$$\int_0^A e^{-st} f'(t)\, \mathrm{d}t = \left[e^{-st} f(t)\right]_0^A + s \int_0^A e^{-st} f(t)\, \mathrm{d}t$$

Main Theorem 2.0

Corollary 6.2.2

1. If $f, f', ..., f^{(n-1)}$ are continuous on $[0, A]$, for any $A > 0$
2. If $|f^{(i)}(t)| \leq Ke^{at}$ for $t > M$ and $i = 0, \ldots, n-1$, with $K, M, a \in \mathbb{R}$ and $K, M > 0$.

Then the Laplace transform $\mathcal{L}\left(f^{(n)}\right)(s)$ exists for $s > a$ and

$$\mathcal{L}\left(f^{(n)}\right)(s) = s^n \mathcal{L}(f)(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Upshot: We can write $\mathcal{L}(f^{(n)})(s)$ in term of $\mathcal{L}(f)(s)$ and the values of $f^{(i)}(0)$.

$$\mathcal{L}\left(f^{(n)}\right)(s) = s^n\mathcal{L}(f)(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Exercise 6.2.11

Use the Laplace transform to solve

$$y'' - y' - 6y = 0; \quad y(0) = 1, y'(0) = -1$$

From Chapter 3, we already know that

$$y(t) = c_1 e^{3t} + c_2 e^{-2t}, \quad c_1 = \frac{1}{5}, c_2 = \frac{4}{5}$$

Upshot of §6.1 and 6.2

nth-order linear (with constant coefficients) ODEs in the t-domain

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = g(t)$$

$$\Leftrightarrow$$

Algebraic equations in the s-domain

$$+$$

Inverting Laplace Transform

Next sections: We will address generalize $g(t)$.

Definition

The function $u_c(t) := \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$ is known as the **unit step function** or **Heaviside function**.

Exercise

Check $\mathcal{L}(u_c)(s) = \begin{cases} e^{-cs}\frac{1}{s}, & c > 0 \\ \frac{1}{s} & c < 0 \end{cases} \quad s > 0$

Indeed,

$$\mathcal{L}(u_c(t)f(t-c))(s) = e^{-cs}\mathcal{L}(f)(s), \quad c > 0$$

$$u_c(t) := \begin{cases} 0, & t < c \\ 1, & t \geq c. \end{cases}$$

Theorem 6.3.1

If $\mathcal{L}(f)(s)$ exists for $s > a \geq 0$ and $c > 0$, then

$$\mathcal{L}\left[u_c(t)f(t-c)\right](s) = e^{-cs}\mathcal{L}(f)(s), \quad s > a$$

Theorem 6.3.2

If $\mathcal{L}(f)(s)$ exists for $s > a \geq 0$, then

$$\mathcal{L}\left[e^{ct}f(t)\right](s) = \mathcal{L}(f)(s-c), \quad s > a + c$$

Exercise 6.3.20

Exercise

Find the inverse Laplace Transform of

$$\frac{e^{-2s}}{s^2+s-2}$$

- $\dfrac{1}{s^2+s-2} = \dfrac{1}{3}\left(\dfrac{1}{s-1} - \dfrac{1}{s+2}\right)$
- $\dfrac{1}{s-a} = \mathcal{L}(e^{at})$
- $e^{-2s}\mathcal{L}(f)(s) = \mathcal{L}(u_2(t)f(t-2))$
- $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2+s-2}\right) = \frac{1}{3}u_2(t)(e^{t-2} - e^{4-2t})$

Exercise 6.2.24

A typical exercise from §6.4.

Exercise 6.2.24

Solve $y'' + 4y = \begin{cases} 1, & 0 \leq t < \pi, \\ 0, & t \geq \pi; \end{cases} \quad y(0) = 1, \quad y'(0) = 0$

Note: $y'' + 4y = 1 - u_\pi(t)$

$$y(t) = \cos(2t) + \frac{1}{4}(1 - \cos(2t))(1 - u_\pi(t)) = \begin{cases} \frac{1+3\cos(2t)}{4} & 0 \leq t < \pi \\ \cos(2t) & t \geq \pi \end{cases}$$

Impulse functions, §6.5

We want a function δ such that:

- $\delta(t) = 0$ for $t \neq 0$
- $\int_{-\infty}^{+\infty} \delta(t) f(t)\, \mathrm{d}t = f(0)$ for f continuous at 0.

There is **no** such function!

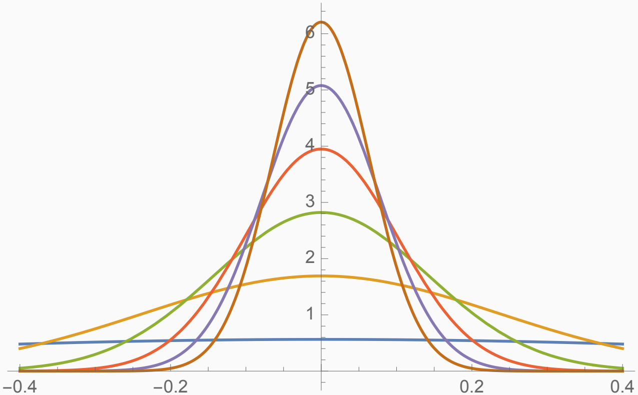
However we can use it as “generalized function“.

$\delta(t)$ is known as **unit impulse function** or as **Dirac delta function**
Even though $\delta(t)$ is **NOT** a function!

$$\mathcal{L}(\delta(t))(s) = 1 \qquad \mathcal{L}(\delta(t-c))(s) = e^{-cs}$$

δ as a non existing limit of functions

We can think of $\delta(t)$ as the limit $\lim_{a\to 0} \frac{1}{a\sqrt{\pi}} e^{-(x/a)^2}$



Click here for gif!

As a non existing derivative

If f is differentiable and $\lim_{t\to+\infty} f(t) = 0$ we have

$$\int_{\mathbb{R}} u_0(t)(-f'(t))\,\mathrm{d}t = \int_0^{+\infty} (-f'(t))\,\mathrm{d}t$$
$$= f(0) - \lim_{A\to+\infty} f(A) = f(0)$$

$$\int_{\mathbb{R}} u_0(t)(-f'(t))\,\mathrm{d}t \text{ `` = ''} \lim_{A\to+\infty} [-u_0(t)f(t)]_{-A}^{A} + \int_{\mathbb{R}} \frac{\mathrm{d}u_0}{\mathrm{d}t}(t)f(t)\,\mathrm{d}t$$
$$= f(0)$$

One can think of $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}u_0(t)$

Click to check:

- $u_0(x)$ on Wolfram Alpha: **link**
- $u_0'(x)$ on Wolfram Alpha: **link**

Exercise 6.5.6

Solve $y'' + 4y = \delta(t-4); y(0) = \frac{1}{2}, y'(0) = 0$

In the s-domain, with $F(s) = \mathcal{L}(y)(s)$

$$s^2F(s) - s\frac{1}{2} - 0 + 4F(s) = e^{-4s}$$

$$\Leftrightarrow$$

$$F(s) = \frac{1}{s^2+4}\left(e^{-4s} + \frac{s}{2}\right) = \frac{1}{2}\left(e^{-4s}\frac{2}{s^2+4} + \frac{s}{s^2+4}\right)$$

$$\Leftrightarrow$$

$$y(t) = \frac{1}{2}\left(u_4(t)\sin(2(t-4)) + \cos(2t)\right)$$

Exercise 6.5.12

Exercise 6.5.12

Solve

$$y^{(4)} - y = \delta(t-1); \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$$

In the s-domain, with $F(s) = \mathcal{L}(y)(s)$

$$s^4 F(s) - F(s) = e^{-s}$$

$$\Leftrightarrow$$

$$F(s) = e^{-s}\frac{1}{s^4-1} = e^{-s}\frac{1}{(s-1)(s+1)(s^2-1)} = \frac{e^{-s}}{4}\left((-2)\frac{1}{s^2+1} - \frac{1}{s+1} + \frac{1}{s-1}\right)$$

$$\Leftrightarrow$$

$$y(t) = \frac{u_1(t)}{4}\left(-e^{1-t} + e^{t-1} + 2\sin(1-t)\right)$$

The convolution integral

Theorem 6.6.1

If $F(s) = \mathcal{L}(f)(s)$ and $G(s) = \mathcal{L}(g)(s)$ for $s > a \geq 0$, then

$$H(s) = F(s)G(s) = \mathcal{L}(h)(s)$$

where

$$h(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s = \int_0^t f(t)g(t-s)\,\mathrm{d}s := (f * g)(t)$$

The function $h(t) = (f * g)(t)$ is known as the **convolution of f and g.**

Exercise 6.6.14

Solve $y'' + 2y' + 2y = \sin(\alpha t); \quad y(0) = 0, y'(0) = 0$

- $\mathcal{L}(\sin(\alpha t))(s) = \frac{\alpha}{s^2+\alpha^2}$
- $s^2F(s) + 2sF(s) + 2F(s) = \frac{\alpha}{s^2+\alpha^2}$
- $F(s) = \frac{1}{(s+1)^2+1}\frac{\alpha}{s^2+\alpha^2}$
- $\mathcal{L}(e^{ct}f(t)) = \mathcal{L}(f)(s-c)$
- $$\begin{aligned} y(t) &= \sin(\alpha t) * \left(e^{-t}\sin t\right) \\ &= \int_0^t \sin((t-z)\alpha)e^{-z}\sin z \, \mathrm{d}z \end{aligned}$$

Check out the Khan Academy video solving the same problem: link

Volterra integral equation

Exercise 6.6.21

Solve

$$\phi(t) + \int_0^t k(t - z)\phi(z)\,\mathrm{d}z = f(t)$$

in terms of $\mathcal{L}(f)$ and $\mathcal{L}(k)$.

Exercise 6.6.25

1. Take $k(t) = 2\cos(t)$ and $f(t) = e^{-t}$, and solve the equation above.
2. Convert the equation above into a 2nd order differential equation

Use: $\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t k(t - z)\phi(z)\,\mathrm{d}z = k(0)\phi(t) + \int_0^t k'(t - z)\phi(z)\,\mathrm{d}z$