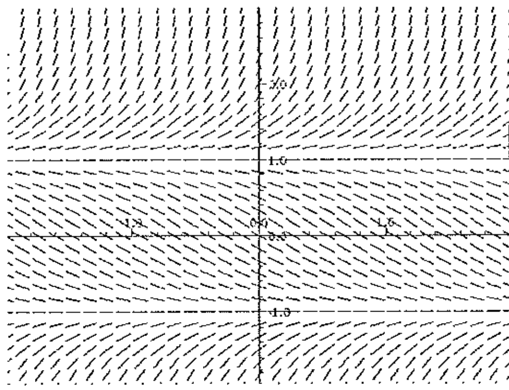
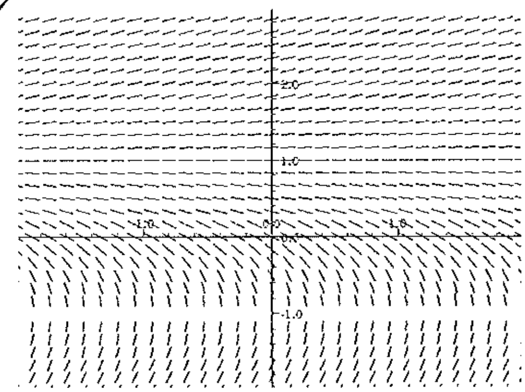
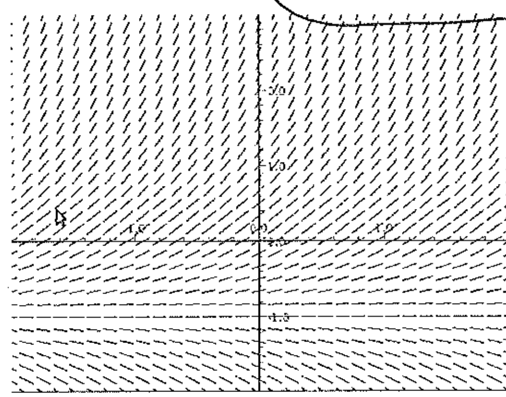
(1) **[10 points]** Consider the differential equation $(y+1)y' = y-1$.

(a) Which of the following is the direction field plot for this equation?







(b) Find all equilibrium (i.e. constant) solutions of this differential equation.

$y' \equiv 0$ when $y = 1$.

Equilibrium: $y = 1$.

(c) Describe the behavior of the solutions as $t \to \infty$.
(Your answer will be in the following form: If the initial value y_0 lies in the interval ______, then y approaches ______ as $t \to \infty$. Consider all possible values of y_0.)

If $y(0) = y_0 > 1$, solution y approaches ∞ as $t \to \infty$.

If $y_0 = 1$, then $y \equiv 1$

(If $y_0 < 1$, the solution is only defined for $t <$ some constant & thus one can't speak of the limit as $t \to \infty$.)

(2) **[12 points]** For each of the following differential equations, indicate (i) whether it is separable, (ii) whether it is linear, and (iii) whether it is exact. You do **not** have to solve the equations. This is a short answer question.

(a) $2xyy' + (e^x + y^2) = 0$

not separable

not linear

exact: $M = e^x + y^2 \quad N = 2xy$

$$\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$$

(b) $t^2 y' + \sin(t)(y - 5) = 0$

separable $\quad \frac{y'}{y-5} = -\frac{\sin t}{t^2}$

linear: $t^2 y' + (\sin t) y = 5 \sin t$

not exact

(3) **[13 points]** Find all solutions of $ty' + 3(t+1)y = \frac{1}{t^2}$, $t > 0$.

linear. Rewrite as

~~$\int (t+1)\,dt$~~

$$y' + \left(3 + \frac{3}{t}\right)y = \frac{1}{t^3}$$

$$\int \left(3 + \frac{3}{t}\right) dt = 3t + 3\ln t + C \quad (\text{since } t > 0)$$

Integrating factor: $e^{3t + 3\ln t} =$

$$= e^{3t} e^{3\ln t} = e^{3t} e^{\ln(t^3)} = t^3 e^{3t}$$

$$\left(t^3 e^{3t} y\right)' = \frac{1}{t^3}\left(t^3 e^{3t}\right) = e^{3t}$$

$$t^3 e^{3t} y = \frac{1}{3} e^{3t} + C$$

$$y = \frac{1}{3t^3} + \frac{C}{t^3 e^{3t}}$$

(4) **[13 points]** Find all solutions of $y' = 2ty^{3/2}$.

Equilibrium sol'n: $y \equiv 0$.

For other sol'ns, separate variables:

$$y^{-3/2}\, y' = 2t$$

$$-2y^{-1/2} = t^2 + C_1$$

$$\cancel{y^{-1/2} = -\frac{1}{2}t^2 - \frac{1}{2}C_1}$$

$$\cancel{y = \frac{4}{(C - t^2)^2}}$$

$$y^{-1/2} = \frac{t^2 + C}{(-2)}$$

$$y = \frac{4}{(t^2+C)^2}$$

(5) **[14 points]** For each of the following differential equations, (i) find the solution of the corresponding homogeneous equation and (ii) indicate the form of a particular solution. You do **not** have to solve for the coefficients.

(a) $y'' + 2y' = \cos(t)$

homog: $r^2 + 2r = 0$ $\quad r(r+2) = 0$ $\quad r = 0, -2$

$y_{hom} = C_1 + C_2 e^{-2t}$

$y_{part} = A\cos(t) + B\sin(t)$

(b) $y'' + 25y = t\cos(5t)$

homog. $r^2 + 25 = 0$ $\quad r = \pm 5i$

$y_{hom} = A\cos(5t) + B\sin(5t)$

If $\pm 5i$ were not roots of the homog eq, our partic sol'n would be of form

$(C_1 + C_2 t)\cos(5t) + (C_3 + C_4 t)\sin(5t)$

Because they are roots, must mult. by t:

$t(C_1 + C_2 t)\cos(5t) + t(C_3 + C_4 t)\sin(5t)$

(6) [12 points]Find all solutions of $ty'' + y' = 0$, $t > 0$.

Let $u = y'$

$t u' + u = 0$ $\qquad$ $t u' = -u$

$$\frac{u'}{u} = -\frac{1}{t}$$

$$\ln |u| = -\ln t + C_1$$
$$= \ln\left(\frac{1}{t}\right) + C_1$$

$$|u| = e^{C_1} \frac{1}{t}$$

$$u = \frac{\pm e^{C_1}}{t} = \frac{C}{t}$$

$$y' = \frac{C}{t}$$

$$y = C \ln t + C_2$$

(7) **[12 points]** A 100 gallon tank of water contains a dye concentration of 1 lb/gal. A solution with a concentration of 2 lbs/gal of dye is added at a rate of 5 gal/min and pure water is added at a rate of one gallon per minute. The well-stirred mixture is draining from the tank at the rate of six gallons per minute. Set up an initial value problem for the amount of dye in the tank at time t. You do **not** have to solve the equation.

y = amt of dye.

$y' =$ (rate in) $-$ (rate out)

$$= 2\,\frac{\text{lb}}{\text{gal}} \cdot 5\,\frac{\text{gal}}{\text{min}} - \frac{y\ \text{lbs}}{100\ \text{gal}}\ \frac{6\ \text{gal}}{\text{min}}$$

$$y' = 10 - \frac{6y}{100} = 10 - .06y$$

(8) [**14 points**] A mass–spring system satisfies the equation

$$u'' + 2u' + u = 0.$$

Suppose that at time zero, the mass is supported so that it is above the equilibrium position (i.e., $u(0) = u_0 < 0$), and then it is released with a push imparting an initial velocity of v_0. Determine a condition that v_0 must satisfy in order to guarantee that the mass passes through the equilibrium point (i.e. the steady state.)

sol'n: $r^2 + 2r + 1 = 0$ $\quad (r+1)^2 = 0 \quad r = -1$

$$u = (C_1 + C_2 t) e^{-t}$$

$$u(0) = u_0 \Rightarrow C_1 = u_0$$

$$u'(t) = (-C_1 - C_2 t + C_2) e^{-t}$$

$$v_0 = u'(0) = C_2 - C_1$$

$$\text{so } C_2 = v_0 + C_1 = v_0 + u_0$$

$$u = [u_0 + (u_0 + v_0) t] e^{-t}$$

Passes thru equilibrium $\Leftrightarrow$ there is a pos. solution $t > 0$ to

$$u_0 + (u_0 + v_0) t = 0.$$

$$t = \frac{-u_0}{u_0 + v_0}$$

$u_0 < 0$ so $-u_0 > 0$. Thus we require that $u_0 + v_0 > 0$.

$$v_0 > -u_0.$$