23F04	Midterm	Exam Time: Wednesday November 3rd, 10.00 - 11.05

Name:	Student No.:

Instructions:

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT seperate the pages of your exam.

Problem	Points	Score
A1	10	
A2	10	
A3	10	
A4	10	
A5	10	
Total	50	

Name:

Section A: Answer ALL questions.

Problem A1: [10 pts] Find an explicit solution to the initial value problem

$$\begin{cases} ty' + 2y = t^2 - 3t \\ \\ y(1) = -1. \end{cases}$$

Solution:

Divide through by t to put the equation in standard form.

$$y' + \frac{2}{t}y = t - 3.$$

Then $p(t) = 2/t$ and the integrating factor is $R(t) = t^2$. The equation then becomes

$$\left(t^2 y\right)' = t^3 - 3t^2.$$

Integrating

$$t^2 y = \frac{t^4}{4} - t^3 + C.$$

Thus the general solution is

$$y = \frac{t^2}{2} - t + Ct^{-2}.$$

Setting $y(1) = -1$ we see that $C = -\frac{1}{4}$ and so the solution to the IVP is

$$y = \frac{t^2}{2} - t - \frac{1}{4t^2}.$$

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Problem A2: [10 pts]   Find the general solution to the equation

$$y'' + y' - 2y = 2e^t.$$

Solution:

The characteristic polynomial for ODE is

$$r^2 + r - 2 = (r+2)(r-1) = 0$$

which has roots $r = 1, -2$. The general solution to the homogeneous equation $y'' + y' - 2y = 0$ is thus

$$y = Ae^t + Be^{-2t}.$$

Since e^t occurs in the homogeneous solution when choosing a form for the particular solution we must guess $Y(t) = Cte^t$. Then $Y' = (C + Ct)e^t$ and $Y'' = (2C + Ct)e^t$. Plugging Y into the equation we then get

$$(2C + C)e^t + (C + C - 2C)te^t = 2e^t.$$

Thus $C = \frac{2}{3}$ and the general solution to the ODE is

$$y(t) = Ae^t + Be^{-2t} + \frac{2}{3}te^t.$$

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Problem A3: [10=8+2 pts] Consider the system of first order linear equations given by

$$\boldsymbol{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \boldsymbol{x}.$$

(**a**) Find the general solution to the differential equation.

Solution:

First we find the eigenvalues

$$\det \begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} = (3-r)(-2-r)+4 = r^2 - r - 2 = (r-2)(r+1).$$

Thus the matrix has eigenvalues -1 and 2.

For $r=-1$, $\begin{pmatrix} 3+1 & -2 \\ 2 & -2+1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$ has eigenvector $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $r=2$, $\begin{pmatrix} 3-2 & -2 \\ 2 & -2-2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$ has eigenvector $\boldsymbol{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The general solution is then

$$\boldsymbol{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}.$$

(**b**) For this system is the critical point at **0** stable, asymptotically stable or unstable?

Solution:

One of the eigenvalues is positive so the critical point is unstable.

Name:

Problem A4: [10 pts] Use the method of variation of parameters to find the general solution to the ODE

$$y'' - 2y' + y = \frac{e^t}{1+t^2}.$$

Solution:

The characteristic polynomial is $r^2 - 1 = 0$ which has a repeated root of $r = 1$. The pair $\{e^t, te^t\}$ is then a fundamental set of solutions.

The method of variation of parameters tells us to construct a solution $y = u_1(t)e^t + u_2(t)te^t$ where u_1 and u_2 satisfy the system

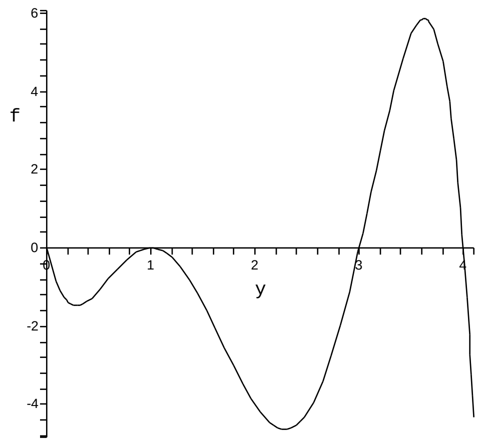
$$\begin{aligned} u_1'e^t + u_2'te^t &= 0 \\ u_1'e^t + u_2'(1+t)e^t &= \frac{e^t}{1+t^2}. \end{aligned}$$

Subtracting the first from second yields that $u_2'e^t = \frac{e^t}{1+t^2}$ and so $u_2' = \frac{1}{1+t^2}$ and $u_2 = \tan^{-1} t + C_2$. Substituting back in we see that $u_1' = -\frac{t}{1+t^2}$ and so $u_1 = -\frac{1}{2}\ln(1+t^2) + C_1$. Thus the general solution is

$$y = C_1e^t + C_2te^t + (\tan^{-1} t)te^t - \frac{1}{2}\ln(1+t^2)e^t.$$

Name:

Problem A5: [10=5+5 pts] The following is a plot of $f(y)$ versus y.



(**a**) Suppose $y(t)$ is a solution to the IVP $\begin{cases} y' = f(y), \\ y(0) = 2. \end{cases}$ Find $\lim_{t\to\infty} y(t)$.

Solution:

The ODE has critical points at $y = 0, 1, 3, 4$. In the range $1 < y < 3$ the function $f(y) < 0$. Therefore this solution $y(t)$ will approach the critical point at $y = 1$. Thus

$$\lim_{t\to\infty} y(t) = 1.$$

(**b**) The ODE $y' = f(y)$ is used to model the population of a species of insect where y is measured in thousands. Since this is a physical situation, the actual size of the population will suffer from small random fluctuations. What size of initial population is needed to prevent eventual extinction of the species?

Solution:

The critical points can be classified as follows. $y = 0$ is stable (it represents extinction and negative y values don't make sense here). $y = 1$ is semi-stable and we would expect random fluctuations to cause solutions near this value to eventually sink below it. $y = 3$ is unstable. $y = 4$ is stable. Any initial population with $y(0) < 3$ will eventually become extinct as it will in the long term cross the semi-stable critical point. $y(0) = 3$ is undetermined - we cannot tell if it will fluctuate up or down. Thus we require $y(0) > 3$