Math 23: Differential Equations (Winter 2017)

Midterm Exam Solutions

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1. *[20 points]* TRUE or FALSE? You do not need to justify your answer.

   (a) *[3 points]* Critical points or equilibrium points for a first order ordinary differential equation $y'(t) = f(t, y)$ are those points where the solution is zero or where the slope of the solution is a constant everywhere.
   FALSE.

   (b) *[3 points]* The ODE $\dfrac{dy}{dt} - ty + t = (y-1)(y-t)$ is autonomous.
   TRUE

   (c) *[3 points]* There is a solution to the ODE $y''+3y'+y = \cos 6t$ of the form $y_p(t) = A\cos 6t$.
   FALSE

   (d) *[3 points]* The differential equation $y'' + t^2y' - y = 3$ is linear.
   TRUE

   (e) *[3 points]* If y_1 and y_2 are two solutions of a nonhomogeneous equation $ay'' + by' + cy = f(x)$, then their difference is a solution of the equation $ay'' + by' + cy = 0$.
   TRUE

   (f) *[5 points]* If $f(x)$ is continuous everywhere, then there exists a unique solution to the following initial value problem.

   $$f(x)y' = y, \quad y(0) = 0$$

   FALSE

2. *[10 points]* If $2xy^3 + 3y\cos(xy) + (cx^2y^2 + 3x\cos(xy))y' = 0$ is an exact equation, what is the value of c?

   Solution: An exact equation is one of form $M(x,y) + N(x,y)y' = 0$ such that $M_y = N_x$. In this example, we let $M = 2xy^3 + 3y\cos(xy)$ and $N = cx^2y^2 + 3x\cos(xy)$, and compute $M_y = 6xy^2 + 3\cos(xy) - 3xy\sin(xy)$ and $N_x = 2cxy^2 + 3\cos(xy) - 3xy\sin(xy)$. For M_y and N_x to be equal, it must be that $c = 3$.

3. *[10 points]* Find the general solution of the differential equation

$$y^{(4)} - 16y = 0$$

   Solution: The characteristic equation here is $r^4 - 16 = 0$. The left hand side factors as $(r^2-4)(r^2+4) = (r+2)(r-2)(r+2i)(r-2i)$, so the roots are $r = 2, -2, 2i, -2i$. Thus, the general solution to the differential equation is

$$y = C_1e^{2t} + C_2e^{-2t} + C_3\cos(2t) + C_4\sin(2t).$$

4. *[10 points]* Find a differential equation with general solution

$$y = C_1e^{-2t} + C_2e^{t} + C_3e^{3t}.$$

   Solution: Working our way backwards, this is the general solution to an ODE with characteristic equation with roots $r = -2, 1, 3$ (with no repetition). So the characteristic equation is $(r+2)(r-1)(r-3) = 0$, i.e. $r^3 - 2r^2 - 5r + 6 = 0$. This corresponds to the homogeneous equation

$$y''' - 2y'' - 5y' + 6y = 0.$$

5. *[15 points]* Solve the differential equation

$$yy' = y^2 + t$$

   by making the substitution $u = y^2 + t$.

   Solution: Since $u = y^2 + t$, we get that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2y\frac{\mathrm{d}y}{\mathrm{d}t} + 1,$$

   or if we just write $u' = \mathrm{d}u/\mathrm{d}t$ and $y' = \mathrm{d}y/\mathrm{d}t$, then $u' = 2yy' + 1$, so $yy' = (u'-1)/2$. Substituting in the differential equation, we get

$$\frac{u'-1}{2} = u,$$

   which in standard form is

$$u' - 2u = 1.$$

   This is a linear first order equation, the integrating factor is $\mu(t) = e^{-2t}$, so the general solution is

$$u = \frac{\int e^{-2t}}{e^{-2t}} = \frac{-\frac{1}{2}e^{-2t} + C}{e^{-2t}} = -\frac{1}{2} + Ce^{2t}.$$

Substituting back, we get

$$y^2 + t = -\frac{1}{2} + Ce^{2t},$$

so

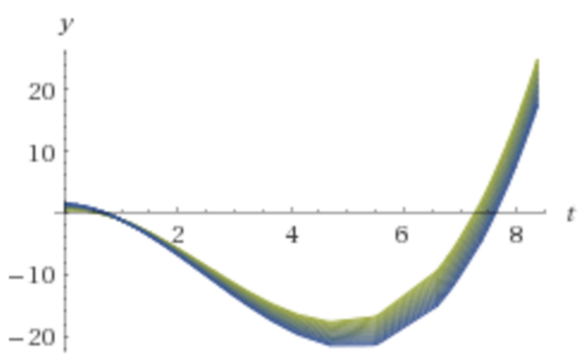
$$y = \pm\sqrt{-\frac{1}{2} + Ce^{2t} - t}.$$

6. *[15 points]* Match the following differential equations to the general solution graphs:

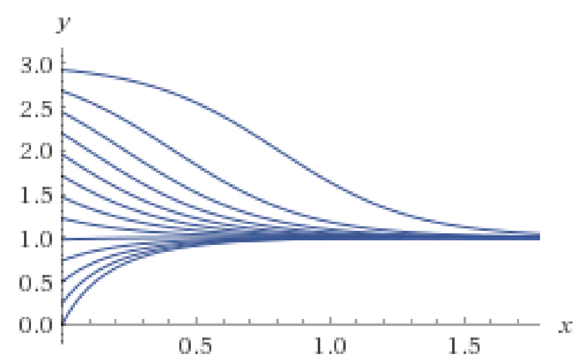
(a) $y' = 2(y-1)(y-3)$ | *B*

(b) $y'' - 2t + 5 = 0$ | *A*

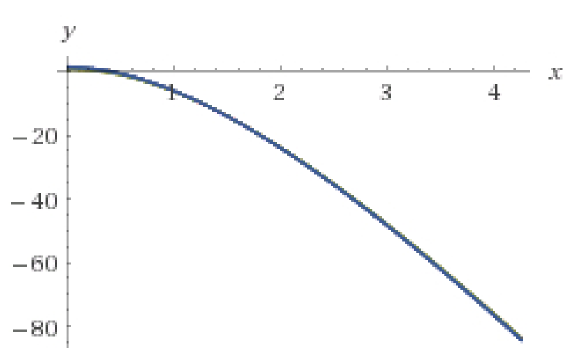
(c) $y'' + \frac{y'}{2} + 7 = 0$ | *C*



A



B



C

7. *[20 points]* Consider the differential equation

$$y'' + by' + 16y = 0$$

(a) For which value(s) of b does the solution

(i) decay rapidly to 0 as $t \to \infty$

(ii) oscillate regardless of t value

(iii) decay while oscillating

Solution:

The roots of the characteristic equation $r^2 + br + 16 = 0$ lead us to the solution of the given ODE. Roots are $r_1 = \frac{-b}{2} + \frac{\sqrt{b^2-64}}{2}$ and $r_2 = \frac{-b}{2} - \frac{\sqrt{b^2-64}}{2}$. Therefore the general solution is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

when $r_1 \neq r_2$, that is when $b \neq 8, -8$. r_1, r_2 may be real or complex. We need to examine the following regions separately

i. $b > 8$: r_1, r_2 real, unequal and < 0. In this case the solution goes to 0 rapidly as $t \to \infty$.

ii. $b < -8$. r_1, r_2 real and unequal and > 0. In this case the solution grows rapidly as $t \to \infty$.

iii. $b = \pm 8$: The discriminant is $= 0$, repeated roots. Solution is given by $y(t) = c_1 e^{\frac{-bt}{2}} + c_2 t e^{\frac{-bt}{2}}$. If $b = 8$ and assuming $c_1, c_2 > 0$ the solution increases until $t = \frac{2}{b} - \frac{bc_1}{2c_2}$ and then goes to 0 rapidly. Depending on the sign of c_1, c_2 the solution may decrease for a bit and then go to ∞ rapidly. If $b = -8$ solution $\to \infty$ as $t \to \infty$.

iv. $-8 < b < 0$: In this case $\frac{-b}{2} > 0$, the discriminant is negative yielding complex values. The solution is of the form $c_1 e^{\frac{-b}{2}t} \cos \alpha t + c_2 e^{\frac{-b}{2}t} \sin \alpha t$. The solution grows while oscillating as $t \to \infty$.

v. $0 < b < 8$; In this case $\frac{-b}{2} < 0$, the discriminant is negative yielding complex values. The solution is of the form $c_1 e^{\frac{-b}{2}t} \cos \alpha t + c_2 e^{\frac{-b}{2}t} \sin \alpha t$. The solution decays to 0 while oscillating as $t \to \infty$.

vi. $b = 0$: r_1, r_2 are complex conjugates of each other. Solution is of the form $c_1 cos\alpha t + c_2 sin\alpha t$. These oscillate forever.

Putting everything together we get

i) For $b \geq 8$ solution $\to 0$ as $t \to \infty$.

ii) When $b = 0$ solution oscillates regardless of t value.

iii) When $0 < b < 8$ solution decays to 0 while oscillating.

(b) For $b = 10$ and $y(0) = 0,\ y'(0) = 6$, solve the initial value problem.

Solution:
Roots of the characteristic equation are $-2, -8$. So general solution $y_c(t)$ is given by $c_1e^{-2t} + c_2e^{-8t}$. Plugging in Initial conditions, we get:

$$y(0) = c_1 + c_2 = 0$$

$$y'(0) = -2c_1 - 8c_2 = 6$$

solving the above simultaneously we get $c_1 = 1,\ c_2 = -1$. Our solution is therefore $y(t) = e^{-2t} - e^{-8t}$.