Math 23, Midterm
May 3, 2017

Name ______________________________ (please print)

Instructor: *Edgar Costa* or *Anne Gelb* (please circle one)

Instructions

- Please **print your name** in the blank space above.
- Please **circle your instructor**.
- Please **turn off cell phones** or other electronic devices which may be disruptive.
- Present your work neatly and clearly. **Justify your answers completely**. Unless explicitly told otherwise, you will not receive full credit for insufficiently justified answers. Please box your answers, when appropriate.
- It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
- This exam is **closed book**. You may not use notes, computing devices (calculators, computers, cell phones, etc.) or any other external resource. It is a violation of the honor code to give or receive help on this exam. However, you may ask the instructor for clarification on problems.
- Consider signing the FERPA waiver:

FERPA waiver: By my signature I relinquish my FERPA rights in the following context: This exam paper may be returned en masse with others in the class and I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructors office to retrieve my examination paper. FERPA waiver signature:

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: ______________________________

Grader's use only:

1. ____________ /24

2. ____________ /15

3. ____________ /20

4. ____________ /20

5. ____________ /15

6. ____________ /15

7. ____________ /16

Total: ____________ /125

1. *[24 points]* TRUE or FALSE? Please keep your justifications short. In some cases, drawing a figure might aid your justification.

   (a) *[4 points]* Given that f is continuous in the equation $y' = f(y)$, it is possible to have 2 unstable equilibrium points with no other equilibrium in between.

   (b) *[4 points]* The function $y(t) = 0$ is never a solution to a linear nonhomogeneous ordinary differential equation.

   (c) *[4 points]* The equation $y' = y - \sqrt{2}t$ has an equilibrium solution.

   (d) *[4 points]* $\{\cos(t), \sin(t)\}$ is a fundamental solution set for $y'' - y = 0$.

   (e) *[4 points]* The ordinary differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} - ty + t = (y-1)(y-t)$ is exact.

   (f) *[4 points]* Given that f is continuous, it is possible to have 2 solutions for the following initial value problem $f(x)y' = y, \quad y(0) = 0$.

2. *[15 points]* Consider the separable equation $y' = -2ty^2$, with the initial condition $y(0) = y_0$.

   (a) Solve the initial value problem.

   (b) Determine the longest interval in which the solution for the above initial value problem exists (which depends on the value of y_0).

3. *[20 points]* Consider the differential equation

$$(x+2)\sin y + (x\cos y)y' = 0.$$

(a) Is the equation above exact? If not, is there an integrating factor which can make it exact?

(b) Find the general solution to the equation.
(Use the integration results $\int xe^x dx = (x-1)e^x + c$ and $\int x^2e^x dx = (x^2-2x+2)e^x + c$ to simplify your answer).

(c) Find the solution satisfying $y(1) = \pi/2$.

4. *[20 points]* Consider the system

$$y'' + by' + y = 0$$

(a) For $b = 2$, suppose that $y(0) < 0$, determine a condition for $y'(0)$ such that $y(t) = 0$ for some $t > 0$.

(b) Determine the value(s) of b such that a nonzero solution $y(t)$ crosses the t-axis once.

(c) Determine the value(s) of b such that a nonzero solution $y(t)$ crosses the t-axis infinitely many times.

(d) Determine the value(s) of b such that a nonzero solution $y(t)$ decays to 0 as $t \to +\infty$.

5. *[15 points]* A mass weighing 32lbs stretches a spring 8 ft. (Note that the mass of the solid is $1\,\text{slug} = 1\,\text{lb}\cdot\text{s}^2/\text{ft}$, with gravity $g = 32\text{ft}/\text{s}^2$.) The mass is initially at rest in its equilibrium position, and there is no damping.

   Suppose that at time $t = 0$ seconds, an external force $f(t) = \cos(2t)\,\text{lb}$ is applied to the solid, but at time $t = \pi$ this force is turned off and the solid is allowed to continue its motion unimpeded. Find the resulting position function $u(t)$ of the solid.

6. *[15 points]* Consider a lake that is stocked with walleye pike (a species of fish) and that the population of pike is governed by the logistic equation

$$P' = .1P(1 - \frac{P}{10})$$

where time is measured in days and P in thousands of fish. Suppose that fishing is started in this lake and that 100 fish are removed each day.

(a) Modify the logistic model to account for the fishing. Write your model in the form $P' = F(P)$. (Hint: Remember that P is measured in *thousands*.)

(b) Find and classify the equilibrium points for your model. That is, determine which points are stable or unstable.

(c) Use qualitative analysis to completely discuss the fate of the fish population with this model. Your analysis should include a sketch of $F(P)$ and the corresponding phase line and direction field. In particular, if the initial fish population is 1000, what happens to the fish as time passes? What will happen to an initial population having 2000 fish? (Hint: again remember that P is measured in *thousands*.)

7. *[16 points]* Match the following differential equations to the direction fields:

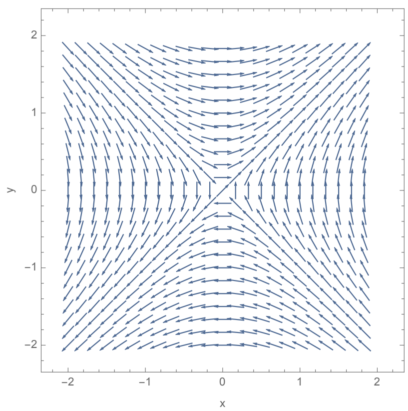
(a) $y' + y = e^{x/3}$

(b) $xy' + 2y = 4x^2$

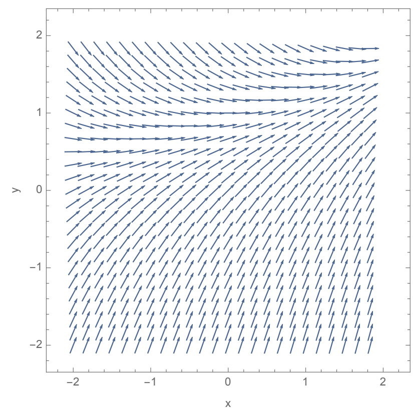
(c) $yy' = -x$

(d) $yy' = x$

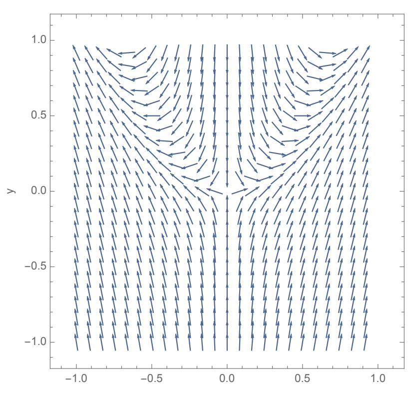
A:



B:



C:



D

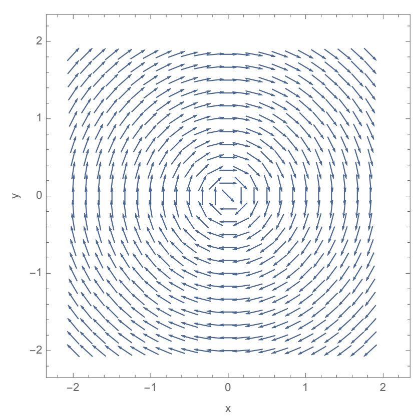


Table 1: Laplace Transform Table.

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$, $s > \|a\|$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s > \|a\|$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$, $s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}

Common integral formulas:

1. $\int \sin^2 u du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
2. $\int \cos^2 u du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
3. $\int u\cos u du = \cos u + u\sin u + C$
4. $\int u\sin u du = \sin u - u\cos u + C$
5. $\int e^{au}\sin bu du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$
6. $\int e^{au}\cos bu du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$