Math 23: Differential Equations
Midterm Exam

January 25, 2018

NAME: ______________________________

SECTION (check one box):

Section 1 (S. Nanda 10:10) ☐
Section 2 (A. Gelb 12:50) ☐
Section 3 (P. Puente 2:10) ☐

Instructions:

1. **Wait** for **signal** to begin.
2. **Write** your **name** in the space provided, and **check one box** to indicate which section of the course you belong to.
3. Please **turn off cell phones** or other electronic devices which may be disruptive.
4. Unless otherwise stated, you must **justify your solutions** to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.
5. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
6. This exam is **closed book**. You may not use notes, or other external resource. You may use calculators. It is a violation of the honor code to give or receive help on this exam. However, you may ask the instructor for clarification on problems.

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: ______________________________

Problem	Points	Score
1	24	
2	20	
3	22	
4	13	
5	21	
Total	100	

1. *[24 points]* TRUE or FALSE? You must provide a *concise* justification if you claim the statement is false. In some cases a counter-example might be the easiest way to justify your answer.

   (a) Critical points (equilibrium points) for a first order ordinary differential equation $y'(t) = f(t, y)$ are those points where the slope of the solution is a constant everywhere.

   (b) The ODE $\dfrac{dy}{dt} - y - t^2 = (y - t)(y + t)$ is autonomous.

   (c) The differential equation $y'' + t^2 y' - y = 3$ is linear.

   (d) If $f(x)$ is continuous everywhere, then there exists a unique solution to $f(x)y' = y, \quad y(0) = 0$.

   (e) $a \sin(t) + b \cos(t)$ is a solution of $y'' + y = 0$ for all values of $a, b \in \mathbb{R}$.

   (f) Euler's method is a quadratic approximation to the solution $y = \phi(t)$ for the initial value problem $y' = f(t, y), \quad y(t_0) = y_0$.

2. [20-points] Consider the equation $y' = f(y)$, where $f(y) = (y-5)(y-2)(y+3)$.

   (a) *[4-points]* Determine all critical points.

   (b) *[7-points]* Draw a graph of $f(y)$ vs y and a phase line diagram.

   (c) *[9-points]* Draw a graph that describes the long term behavior of the solution for all values $y_0 = y(0)$ and identify equilibrium solutions on the graph. Characterize the equilibrium solutions in terms of their stability.

3. *[22- points]* A tank with a capacity of 100 gallons contains 50 gallons of a solution consisting of water and dye. Initially there is 1 ounce of dye in the solution. Water containing a concentration of $\frac{e^{t/50}}{t+50}$ oz/gal flows in at a rate of 3 gal/min and flows out at 2 gal/min. Assume that the tank is well mixed.

   (a) *[8-points]* Write a differential equation that determines the amount of dye in the solution.

   (b) *[10 -points]* Find the amount of the dye in the solution at any given time t.

   (c) *[4-points]* Find the amount of dye in the solution when the tank is about to overflow.

4. *[13-points]* Find the general solution to

$$e^x + 2xy^2 + (2x^2y - \frac{1}{y})\frac{dy}{dx} = 0$$

5. *[21-points]* Consider the initial value problem

$$y' = y - t^2 + 1, \qquad y(0) = .5, \qquad 0 \le t \le 2.$$

(a) *[2-points]* Verify that $\phi(t) = (t+1)^2 - \frac{e^t}{2}$ is the exact solution for this problem.

(b) *[6-points]* If the step size is given by $h = .2$, determine the local truncation error valid on the interval $[0, 2]$ for solving this problem using the forward Euler's method.

(c) *[4-points]* What step size is needed to ensure that the local truncation error is no greater than .001?

(d) *[9-points]* Draw a diagram that demonstrates the numerical approximation using both forward and backward Euler's method from the point $t_0 = 0$ to the point $t_1 = .2$. You may approximate $\phi(t)$ to simplify your sketch. *Label* the solution at the corresponding values y_0 and y_1.