Second Order Linear ODEs, Part I

Craig J. Sutton
craig.j.sutton@dartmouth.edu

Department of Mathematics
Dartmouth College

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Outline

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The Definition

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A second order ODE has the form

$$\frac{d^2y}{dt} = f(t, y, \frac{dy}{dt}).$$

A second order linear ODE has the form

$$y'' + p(t)y' + q(t)y = g(t)$$

or

$$R(t)y'' + P(t)y' + Q(t)y = G(t).$$

We'll say the equation is homogeneous if $g(t) = 0$ or $G(t) = 0$.

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A Second Order ODE Example: the Vibrating Spring

- Consider a spring with spring constant k and a block of mass m attached to the end.
- Let $x(t)$ denote the displacement of the block-spring system from the spring-mass equilibrium.
- $x(t)$ is governed by the 2nd order linear ODE

$$mx'' = -kx + mg.$$

- Getting fancy we obtain

$$mx'' = -kx + mg + D(x') + F(t),$$

where D is the damping force and $F(t)$ is the external force.

Linearity

If $y_1(t)$ and $y_2(t)$ solve the homogeneous 2nd order linear ODE

$$y'' + p(t)y' + q(t)y = 0$$

then for any c_1 and c_2 the function

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t)$$

solves the ODE.

Initial Value Problems

Definition

A second order linear IVP consists of a 2nd order linear ODE

$$y'' + p(t)y' + q(t)y = g(t) \tag{1.1}$$

and initial conditions

$$y(t_0) = y_0 \text{ and } y'(t_0) = y'_0.$$

Note: There is no general solution method for 2nd order linear ODEs, but we do have an existence and uniqueness theorem.

Existence & Uniqueness

Theorem

Consider the IVP

$$y'' + p(t)y' + q(t)y = g(t), \; y(t_0) = y_0, y'(t_0) = y'_0, \qquad (1.2)$$

where p, q and g are continuous on some open interval I containing t_0. Then there is exactly one solution $y = \phi(t)$ of Eq. 1.2 and it is defined and at least twice differentiable throughout the interval I.

Moral

If p, q and g are continuous, then a solution $\phi(t)$ to the second order linear ODE is uniquely determined by the initial data: $\phi(t_0)$ and $\phi'(t_0)$.

Exercises

1. Find the longest interval on which a solution to the IVP

$$(t^2+7t)y''+(t^3+t)y'-(t+3)y=0,\ \ y(2)=-7, y'(2)=1,$$

is guaranteed to exist.

2. Find a solution to the IVP

$$y''+p(t)y'+q(t)y=0,\ \ y(t_0)=0, y'(t_0)=0,$$

where p and q are continuous on an open interval I containing t_0. Is it unique?

Outline

Game Plan

- Consider 2nd order homogeneous linear ODE

$$y'' + p(t)y' + q(t)y = 0,$$

- We will see that the solutions will come in a $2D$-family

$$c_1 y_1(t) + c_2 y_2(t)$$

where c_1 and c_2 are constants.
- Find a solution $\Psi(t)$ to the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t).$$

- Then all solutions to $y'' + p(t)y' + q(t)y = g(t)$ will be of the form

$$\Psi(t) + c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are arbitrary constants (determined by initial conditions).

Game Plan

- We will restrict our attention to constant coeff. equations:

$$ay'' + by' + cy = g(t)$$

- In the third part of these notes we will see that you can always find explicit solutions to the associated homogeneous problem

$$ay'' + by' + cy = 0$$

- Method of Undetermined Coefficients and Variation of Parameters will help with the non-homogeneous problem.

Moral

Analyze the homogeneous case before tackling the general case.

Example 1

- Consider the IVP

$$y'' - y = 0, \;\; y(0) = 1, y'(0) = 2. \tag{2.1}$$

- $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are distinct solutions to the homogeneous ODE $y'' - y = 0$.
- In fact, $\phi(t) = c_1 e^t + c_2 e^{-t}$ solves the ODE for any choice of c_1 and c_2.
- Can we choose c_1 and c_2 such that $\phi(0) = 1$ and $\phi'(0) = 2$?
- Yes, take $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$
- Then $\phi(t) = \frac{3}{2}e^t - \frac{1}{2}e^{-t}$ solves our IVP.

Example 1 (cont'd)

- Consider the IVP

$$y'' - y = 0, \ \ y(0) = a, y'(0) = b. \tag{2.2}$$

- Can we choose c_1 and c_2 such that

$$\phi(t) = c_1 e^t + c_2 e^{-t}$$

solves the IVP?
- Yes, take $c_1 = \frac{a+b}{2}$ and $c_2 = \frac{a-b}{2}$
- So, any solution to our ODE is of the form

$$c_1 e^t + c_2 e^{-t} \text{ (why?)}$$

That is, we have a two-dimensional family of solutions.

Example 2

- Consider the IVP

$$y'' - 2y' - 35y = 0, \; y(0) = 1, y'(0) = 2. \qquad (2.3)$$

- $y_1(t) = e^{-7t}$ and $y_2(t) = e^{5t}$ are distinct solutions to the homogeneous ODE $y'' - 2y' - 35y = 0$.
- In fact, $\phi(t) = c_1 e^{-7t} + c_2 e^{5t}$ solves the ODE for any choice of c_1 and c_2.
- Can we choose c_1 and c_2 such that $\phi(0) = 1$ and $\phi'(0) = 2$?
- Yes, take $c_1 = \frac{1}{4}$ and $c_2 = \frac{3}{4}$
- Then $\phi(t) = \frac{1}{4} e^{-7t} + \frac{3}{4} e^{5t}$ solves our IVP.

Example 2 (cont'd)

- Consider the IVP

$$y'' - 2y' - 35y = 0, \; y(0) = a, y'(0) = b. \tag{2.4}$$

- Can we choose c_1 and c_2 such that

$$\phi(t) = c_1 e^{-7t} + c_2 e^{5t}$$

solves the IVP?
- Yes, take $c_1 = \frac{5a-b}{12}$ and $c_2 = \frac{7a+b}{12}$
- So any solution to our ODE is of the form

$$c_1 e^{-7t} + c_2 e^{5t} \text{ (why?)}$$

That is, we have a two-dimensional family of solutions.

Moral

In each of the previous examples we were able to find two solutions y_1 and y_2 of our ODE such that the matrix

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$$

is invertible (i.e., has non-zero determinant).
We could then express any solution to the ODE as a linear combination

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t).$$

Question

Does this work for a general 2nd order linear homogeneous ODE?

Linear Independence

Definition

Let f and g be two functions defined on some open interval $I : \alpha < t < \beta$. We will say that f and g are **linearly dependent** on the interval I if there are constants c_1 and c_2 (not both zero) such that

$$c_1 f(t) + c_2 g(t) = 0$$

for all t in the interval I. That is, one of the functions is a scalar multiple of the other. Otherwise, we say that the functions are **linearly independent** on the interval I.

Linear Independence: some Examples

1. $f(t) = t^2 + 2t$ and $g(t) = 0$ are linearly dependent on $-\infty < t < \infty$.
2. $f(t) = t^2 + 2t$ and $g(t) = -9t^2 - 18t$ are linearly dependent on $-\infty < t < \infty$.
3. $f(t) = \cos(t)$ and $g(t) = \sin(t)$ are linearly independent on $-\infty < t < \infty$.
4. $f(t) = \frac{1}{t}$ and $g(t) = \sin(t)$ are linearly independent on $0 < t < +\infty$.

The Wronskian

Definition

Let f and g be two functions defined on some open interval $I : \alpha < t < \beta$. The **Wronskian** of f and g denoted $W(f,g)(t)$ is the function on I defined by

$$W(f,g)(t) = \det \begin{pmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{pmatrix} = f(t)g'(t) - f'(t)g(t).$$

The Wronskian provides a test for linear independence...

The Wronskian & Linear Independence

Theorem

Let f and g be differentiable functions on some interval I. If $W(f, g)(t_0) \neq 0$ for some t_0 in I, then f and g are linearly independent.

Fundamental Sets & the Wronskian

Definition

Two solutions $y_1(t)$ and $y_2(t)$ of the 2nd order linear ODE

$$y'' + p(t)y' + q(t)y = 0, \ \alpha < t < \beta,$$

are said to form a **fundamental set of solutions** (on the interval) if there is a number $\alpha < t_0 < \beta$ such that

$$W(y_1, y_2)(t_0) \neq 0.$$

Question

Does a fundamental set of solutions always exist?

Fundamental Sets & the Wronskian

Question

What's so special about fundamental sets of solutions?

We will see that a fundamental sets of solutions $\{y_1, y_2\}$ to a 2nd Order linear homogeneous ODE on an interval I generate all solutions to the ODE on I.

The Wronskian & Linear Independence

Theorem (3.2.6, Abel's Theorem)

If y_1 and y_2 are solutions to the 2nd order ODE $y'' + p(t)y' + q(t)y = 0$, where p and q are continuous on I, then

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)\, dt\right),$$

where c is a constant that only depends on y_1 and y_2. So, $W(y_1, y_2)(t)$ is zero everywhere on I or never zero.

The Wronskian & Fundamental Sets of Solutions

Theorem

Suppose that $y_1(t)$ *and* $y_2(t)$ *are solutions to the Linear ODE*

$$y'' + p(t)y' + q(t)y = 0.$$

Now suppose

1. p *and* q *are continuous at* t_0
2. $W(y_1, y_2)(t_0) \neq 0$ *(i.e.,* y_1 *and* y_2 *are lin. indep.)*

Then there exist unique constants c_1 *and* c_2 *such that* $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$ *satisfies the IVP*

$$y'' + p(t)y' + q(t)y = 0, \; y(t_0) = y_0, y'(t_0) = y_0'.$$

That is $\{y_1, y_2\}$ *forms a fundamental set of solutions.*

The Wronskian & Fundamental Sets of Solutions

Moral

If $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions to the 2nd Order Linear Homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0$$

on the interval I. Then every solution to this ODE on I can be expressed as

$$c_1 y_1(t) + c_2 y_2(t)$$

for some unique choice of real numbers c_1 and c_2.

Existence of Fundamental Sets

Theorem (3.2.5)

Consider the 2nd order linear ODE

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on some interval I. Choose a point t_0 in I. Let $y_1(t)$ be the solution to the ODE with initial data $y_1(t_0) = 1$ and $y_1'(t_0) = 0$, and let $y_2(t)$ be the solution to the ODE with initial data $y_2(t_0) = 0$ and $y_2(t_0) = 1$. Then $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions for the ODE on the interval I.

All of the Solutions

Theorem

Let $\Psi(t)$ be some solution to the 2nd order linear ODE

$$y'' + p(t)y' + q(t)y = g(t), \ \alpha < t < \beta,$$

where p and q are continuous. Let $\{y_1(t), y_2(t)\}$ be a fundamental set of solutions to the associated homogeneous equation. Then all solutions to our ODE on the interval $\alpha < t < \beta$ are of the form

$$\Psi(t) + c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are constants.

Outline

The Idea

We showed $\{y_1(t) = e^t, y_2(t) = e^{-t}\}$ is a fundamental set of solutions for $y'' - y = 0$. How did we get these solutions?

- Assume solution looks like $y(t) = e^{rt}$.
- Then $y(t)$ solves equation if and only if $r^2 - 1 = 0$. Why?
- Hence, $y(t)$ solves ODE if and only if $r = \pm 1$.
- So e^t and e^{-t} are solutions.
- $W(e^t, e^{-t})(t) \neq 0$ implies fundamental set.

Moral: We reduced solving this const. coeff. equation to finding roots of a quadratic.

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In General

Consider that 2nd order homogeneous linear ODE

$$ay'' + by' + cy = 0.$$

- Assume solution looks like $y(t) = e^{rt}$.
- Then $y(t)$ solves equation if and only if $ar^2 + br + c = 0$.
- Let $\Delta = b^2 - 4ac$.
- There are three cases for the roots r_1 and r_2 of the characteristic equation.
  1. $r_1 \neq r_2$ are real (i.e., $\Delta > 0$).
  2. $r_1 = \nu + i\mu, r_2 = \nu - i\mu$, where $\mu \neq 0$ (i.e., $\Delta < 0$).
  3. $r_1 = r_2$ real numbers (i.e., $\Delta = 0$).

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  3. $r_1 = r_2$ real numbers (i.e., $\Delta = 0$).

Case 1: $\Delta > 0$ (Distinct Real Roots)

- $r_1 \neq r_2$ real roots
- $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ solve

$$ay'' + by' + cy = 0.$$

- $W(y_1, y_2)(t) \neq 0$
- So $\{y_1(t), y_2(t)\}$ forms a fundamental set of solutions
- $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$ is the general solution.

Case 1: $\Delta > 0$ (Distinct Real Roots)

Solve the following IVPs

1. $y'' + 9y' + 20y = 0, \;\; y(1) = 3, y'(1) = 0.$
2. $2y'' - 20y' + 42y = 0, \;\; y(2) = 3, y'(2) = 5.$
3. $3y'' + 12y' + 9y = 0, \;\; y(-1) = 9, y'(-1) = 2.$

Case 2: $\Delta < 0$ (Imaginary Roots)

First we recall a few things

- $i = \sqrt{-1}$
- $z = \nu + i\mu$ is a complex number
- (Euler's Formula) $e^{it} = \cos(t) + i\sin(t)$
- $e^{\nu + i\mu} = e^{\nu}e^{i\mu} = e^{\nu}(\cos(\mu) + i\sin(\mu))$
- If z and w are complex numbers then $e^{z+w} = e^{z}e^{w}$.
- $y(t) = e^{zt}$, then $y'(t) = ze^{zt}$.

Case 2: $\Delta < 0$ (Imaginary Roots)

Now since $\Delta < 0$ we have:

- $r_1 = \nu + i\mu$ and $r_2 = \nu - i\mu$ with $\mu \neq 0$.
- $\tilde{y}_1(t) = e^{r_1 t}$ and $\tilde{y}_2(t) = e^{r_2 t}$ solve

$$ay'' + by' + cy = 0.$$

- $W(\tilde{y}_1, \tilde{y}_2)(t) \neq 0$
- So $\{\tilde{y}_1(t), \tilde{y}_2(t)\}$ forms a fundamental set of solutions
- But $\tilde{y}_1$ and $\tilde{y}_2$ are not real-valued functions.

Case 2: $\Delta < 0$ (Imaginary Roots)

However

- $y_1(t) = \frac{1}{2}\tilde{y}_1(t) + \frac{1}{2}\tilde{y}_2(t) = e^{\nu t}\cos(\mu t)$
- $y_2(t) = \frac{1}{2i}\tilde{y}_1(t) - \frac{1}{2i}\tilde{y}_2(t) = e^{\nu t}\sin(\mu t)$

are real-valued solutions. In fact,

$$W(y_1, y_2)(t) \neq 0.$$

Hence, $\{e^{\nu t}\cos(\mu t), e^{\nu t}\sin(\mu t)\}$ is a fundamental set of solutions.

Case 2: $\Delta < 0$ (Imaginary Roots)

Solve the following IVPs:

1. $y'' + 16y = 0,\ y(\pi/2) = 1, y'(\pi/2) = 0.$
2. $2y'' + 2y' + y = 0,\ y(0) = 1, y'(0) = 3.$
3. $3y'' + 4y' + 3y = 0,\ y(-\pi/4) = 3, y'(-\pi/4) = 2.$

Case 3: $\Delta = 0$ (Repeated Real Roots)

- $r_1 = r_2 = -\frac{b}{2a}$ are real.
- $y_1(t) = e^{r_1 t}$ is a solution, but we need another.
- Suppose $y(t) = v(t)y_1 = v(t)e^{-bt/2a}$ is a solution.
- Then

$$y_2'(t) = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

and

$$y_2''(t) = v''(t)e^{-bt/2a} - \frac{b}{a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}.$$

Case 3: $\Delta = 0$ (Repeated Real Roots)

- Use the differential Eq. to see
$$v''(t) = 0.$$
- Hence $v(t) = c_1 t + c_2$
- $y(t) = c_1 t e^{-bt/2a} + c_2 e^{-bt/2a}$.
- $W(e^{-bt/2a}, te^{-bt/2a})(t) = e^{-bt/2a}$
- Hence, $\{e^{-bt/2a}, te^{-bt/2a}\}$ forms a fundamental set of solutions our ODE.
- This is an example of the method of Reduction of Order

Case 3: $\Delta = 0$ (Repeated Real Roots)

Solve the following IVPs:

1. $y'' - y' + 0.25y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{3}$.
2. $2y'' + 8y' + 8y = 0,\ y(1) = 3\ y'(1) = -2$.
3. $y'' - 6y' + 9y = 0,\ y(-2) = 3\ y'(-2) = 3$.

Case 3: $\Delta = 0$ (Repeated Real Roots)

Here's another way to find the fundamental set of solutions in this case:

- Let $L = a\frac{d^2}{dy^2} + b\frac{d}{dy} + c$ and assume $ar^2 + br + c = 0$ has a repeated root r_1.
- Then
$$L(e^{rt}) = a(r - r_1)^2 e^{rt}$$
it equals zero if and only if $r = r_1$.
- So, as before, $y_1(t) = e^{r_1 t}$ is a solution.

Case 3: $\Delta = 0$ (Repeated Real Roots)

- Now
$$\frac{\partial}{\partial r}L(e^{rt}) = L(\frac{\partial}{\partial r}e^{rt}) = L(te^{rt}) = ate^{rt}(r-r_1)^2 + 2ae^{rt}(r-r_1).$$
- Conclude that $te^{r_1 t}$ is a solution to $L(y) = 0$.
- So, $y_1(t) = e^{r_1 t}$ and $y_2(t) = te^{r_1 t}$ are solutions.
- As before, by computing the Wronskian, we see they form a fundamental set of solutions.
- See Problems 3.4.20 & 3.4.21 for additional ways of seeing this.