Name:

Question you don't want graded:

Math 23 Diff Eq: Final

3 hours, each question worth 15 points. Answer 9 out of 10 questions. You MUST indicate which one you don't want graded above ("whichever is lowest" is not acceptable!). 1 sheet of notes allowed. No calculator needed; no algebra-capable ones allowed. Read questions carefully so you don't miss parts. Good luck and enjoy!

1. Consider $ty' + y = \dfrac{t}{2-t}$

   (a) [8 points] Find the general solution.

   (b) [4 points] Find the solution given $y(1) = 2$

   (c) [3 points] Over what range of t must this solution exist and be unique? (explain your answer)

2. A mass of 1 kg is held by spring of constant 5 N/m with damping coefficient $\gamma = 4$ kg/s. [Note: you may safely ignore all the units]

   (a) [3 points] Is the system under-, over-, or critically-damped? Give a rough sketch of the motion $y(t)$ given typical (nonzero) initial conditions.

   (b) [9 points] A time-dependent force $g(t) = t + \sin t$ is applied to the mass. Find the general solution $y(t)$.

   (c) [3 points] If instead the force were $g(t) = e^{-2t} \sin t$, what form of particular solution would you need? Explain why. (Do not solve; this would take you too long).

3. Consider $y'' + 2y' + y = g(t)$

(a) [5 points] When $g(t) = 0$, that is, it's a homogeneous equation, find a fundamental set of solutions.

(b) [5 points] Given a general nonzero function $g(t)$, write a formula for a *particular solution* (this will be in terms of $g(t)$).

(c) [5 points] Using this find the *general solution* in the case where $g(t) = \dfrac{e^{-t}}{t^2}$.

4. Consider $y'' + xy' + 2y = 0$.

(a) [13 points] Using a power series about $x_0 = 0$, find the general solution. Include the first *four* non-zero terms for each linearly-independent solution. [Hint: make sure to write the recurrence relation, and notice a cancellation].

(b) [2 points] What is the radius of convergence of your series?

5. Consider $y'' + 4y = 0$

(a) [3 points] Write down the general solution $y(t)$.

(b) [5 points] If $y'(0) = 0$ and $y'(\pi/2) = 0$, discuss *existence* and *uniqueness* of any solution $y(t)$ in $0 \leq t \leq \pi/2$. If any exists, give it and sketch it.

(c) [4 points] Repeat the above except given instead $y(0) = 1$ and $y(\pi/2) = 2$.

(d) [3 points] For what values of λ does the boundary value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$ have a *non-unique* solution?

6. Solve the system of equations for $x(t)$ and $y(t)$,

$$\begin{aligned} x' &= 5x - y \\ y' &= 3x + y \end{aligned}$$

subject to the initial conditions $x(0) = 4$, $y(0) = -2$. Show all working. [Hint: the numbers come out nice so stop and check if they're not for you]

7. Consider the nonlinear system

$$\begin{aligned} x' &= -y + y^3 \\ y' &= x \end{aligned}$$

(a) [4 points] Find the *linearized system* at the critical point $(0, 0)$. (That is, find the matrix A in $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x})$). Sketch the form of trajectories of this linearized system.

(b) [2 points] Is this linearized system stable? Asymptotically stable?

(c) [2 points] Categorize (describe) the critical point $(0, 0)$ of the nonlinear system. What can you deduce about its stability?

(d) [4 points] Repeat part (a), including sketch, for the critical point (0,1).

(e) [3 points] Categorize the critical point $(0, 1)$ of the nonlinear system. What can you deduce about its stability? [3 point BONUS: add correct eigenvector directions to your sketch]

8. Consider the function $f(x) = x$ in the interval $[0, \pi)$.

(a) [3 points] Sketch 3 periods of the function produced by extending f as an *odd* (anti-symmetric) function with period 2π.

(b) [7 points] Find the *Fourier sine series* for f in the interval $[0, \pi)$. Try to write your answer as simply as possible.

(c) [3 points] Write the first three nonzero terms in this series.

(d) [2 points] Sketch 3 periods of the function to which the above sine series converges, indicating differences from (a), if any.

9. Consider a rod of length π with temperature distribution $u(x,t)$ evolving according to $u_t = \alpha^2 u_{xx}$. The ends are *insulated* (no heat flux), that is, the boundary conditions are $\partial u/\partial x = 0$ at $x = 0$ and at $x = \pi$, for all time. Leave α as a general constant.

(a) [13 points] Find $u(x,t)$ given the initial condition

$$u(x,0) = \begin{cases} 1, & 0 \le x \le \pi/2 \\ 0, & \pi/2 < x \le \pi \end{cases}$$

(b) [2 points] What is the temperature distribution as $t \to \infty$?

10. Imagine $u(x, y)$ satisfies boundary conditions $u(x, 1) = f(x)$, where f is some given function for $0 < x < 1$, and u vanishes on the other three sides of the unit square, that is $u(x, 0) = 0$, $u(0, y) = 0$, and $u(1, y) = 0$.

   (a) [10 points] If inside the square, u obeys $u_{xx} + u_{yy} = 0$, find the solution (you should express your answer in terms of $f(x)$).

   (b) [5 points] Repeat the above except with u instead obeying $u_{xx} + u_{yy} + 5u = 0$ (the *Helmholtz Equation*). [Hints: try and arrange the x-eigenvalues to be the same as above. Are these eigenvalues greater or less than the number 5?]

[3 point BONUSes: what would happen if the number 5 were replaced by π^2 ? By $2\pi^2$?]