Problem 1. (a) Find the general solution of the differential equation

\[ 2y'' + 3y' + y = \sin 2t \]

(b) What is the behavior of the solution as \( t \to \infty \)?

Solution. The characteristic equation for the corresponding homogeneous equation is \( 2r^2 + 3r + 1 = 0 \), with roots \( r_1 = -1/2, \; r_2 = -1 \). So the general solution to the homogeneous equation is

\[ y = C_1 e^{-t/2} + C_2 e^{-t}. \]

We guess \( Y = A \sin 2t + B \cos 2t \) as a particular solution to the nonhomogeneous equation. Plugging \( Y \) into the nonhomogeneous equation, we get

\[(−7A − 6B) \sin 2t + (6A − 7B) \cos 2t = \sin 2t \]

Solving the system

\[-7A − 6B = 1 \]
\[6A − 7B = 0 \]

we get \( A = -7/85 \) and \( B = -6/85 \), so the general solution to the equation is

\[ y = C_1 e^{-t/2} + C_2 e^{-t} − \frac{7}{85} \sin 2t − \frac{6}{85} \cos 2t. \]

As \( t \to \infty \), for any choice of \( C_1 \) and \( C_2 \) the first two summands of \( y \) approach zero, and the sum of the other two summands oscillates with a constant amplitude, so \( y \) oscillates without approaching a limit as \( t \to \infty \).
Problem 2. (a) Find the general solution of the differential equation

\[ 2y'' - 3y' - y = t^2 \]

(b) What is the behavior of the solution as \( t \to \infty \)?

Solution. The solutions to the characteristic equation are \( r_1 = \frac{3}{4} + \frac{\sqrt{17}}{4} \) (positive) and \( r_2 = \frac{3}{4} - \frac{\sqrt{17}}{4} \) (negative), so the general solution to the corresponding homogeneous equation is \( C_1 e^{r_1 t} + C_2 e^{r_2 t} \). The right hand side of the equation is a polynomial of degree two, and there are no polynomial solutions to the homogeneous equation (so we don’t need to introduce a factor of \( t^s \)), so we try \( Y = At^2 + Bt + C \) as a particular solution. Plugging \( Y \) into the equation, we get

\[ 4A - 6At - 3B - At^2 - Bt - C = t^2, \]

so \( A = -1, B = 6, C = -4 \). Thus, the general solution is

\[ y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} - t^2 + 6t - 4. \]

As \( t \to 0 \), the second summand tends to 0, and the polynomial part to \(-\infty\). When the constant \( C_1 \) is chosen positive, the first term approaches \( \infty \) and dominates the polynomial summand, so \( y \to \infty \). When \( C_1 \leq 0 \), the solution approaches \(-\infty\).

Problem 3. What is the integrating factor \( \mu(t) \) used to solve the first order linear equation:

\[ 2t^2y' - 6ty = e^{-t} \]

Solution. We first need to arrange that the coefficient of \( y' \) is 1. So we divide by \( 2t^2 \) and rewrite the equation as

\[ y' - \frac{3}{t}y = \frac{e^{-t}}{2t^2}. \]

The integrating factor then is

\[ \mu(t) = e^{\int \frac{3}{t}dt} = e^{-3\ln t} = t^{-3}. \]

Problem 4. Solve the initial value problem

\[ y''' - 3y'' + 4 = 0; \quad y(0) = 1, y'(0) = 0, y''(0) = -1 \]

Solution. Use reduction of order. Set \( u = y'' \) and solve \( u' - 3u = -4 \).

Problem 4’. Solve the initial value problem

\[ y''' - 3y'' + 4y = 0; \quad y(0) = 1, y'(0) = 0, y''(0) = -1 \]

Solution. The characteristic equation factors as \((r + 1)(r - 2)^2 = 0\), so the general solution is \( C_1 e^{-t} + C_2 e^{2t} + C_3 t e^{2t} \).

Problem 5. Determine the form of the particular solution when the method of undetermined coefficients is used to solve

\[ y'' - 4y = e^{-2t} + 2t^2 - 1 \]

Do NOT solve for the coefficients.
Solution. Since the right hand side is a sum of a polynomial and an exponential function, we know that we can find a particular solution of the same form, except that we may need to multiply each summand by $t$ or $t^2$ if the characteristic equation for the corresponding homogeneous equation has a repeated root. Since the characteristic equation is $r^2 - 4 = 0$, with roots $\pm 2$, we see that need to multiply by a power of $t$. Thus, the form of the general solution is

$$Y = Ate^{-2t} + Bt^2 + Ct + D.$$
Problem 5. Each differential equation matches a graph of a general solution. Give the letter of the correct graph:

(a) $2y' + 5y = 6$  
(b) $y'' - t = -2$  
(c) $y' = \sqrt{y}$

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![Graphs A, B, C]

Problem 7. Show that every solution of the equation $y' = x^3(y^2 + 1)$ has at most one minimum point.

Solution. If $y$ has a minimum at some $x_0$, then it must be that $y'(x_0) = 0$, i.e. $x_0^3(y(x_0)^2+1) = 0$. This can only happen for $x_0 = 0$, so $y$ either attains a minimum at 0 or nowhere.

Problem 8. Solve the differential equation

$y''' = ty''$

Solution. Using reduction of order, we let $u = y''$ and solve $u' = tu$, which is a first order linear equation. We get $u = Ce^{t^2/2}$, so $y' = \int_{t_0}^{t} Ce^{s^2/2} ds + D$, so

$$y = \int_{t_0}^{t} y'(s) ds + E = \int_{t_0}^{t} \left( \int_{t_0}^{s} Ce^{q^2/2} dq + D \right) ds + E.$$  

In this case, $u$ is not very nice to integrate, so you do not need to simplify your answer further.