MATH 23: DIFFERENTIAL EQUATIONS
WINTER 2017
PRACTICE PROBLEMS FOR FINAL EXAM

Problem 1. TRUE or FALSE?

- (a) e^{rx} is a solution of the equation : FALSE
$$x^2y'' + x\alpha y' + \beta y = 0$$
- (b) If A is an $n \times n$ matrix and $\mathbf{x}', \mathbf{x}$ are n-vectors, then $\mathbf{x}' = A\mathbf{x}$ is a homogeneous system of first order differential equations. TRUE
- (c) If $f(x)$ is continuous on a domain D, then there is a unique Fourier series that converges to f on D. FALSE
- (d) The function $\sin(x - \frac{\pi}{4})$ is odd. FALSE
- (e) The function $e^{|x|}\cos(x^3)$ is even. TRUE

Problem 2. For each of the following systems of equations, find the eigenvalues and corresponding eigenvectors, find the general solution, and sketch a phase portrait:

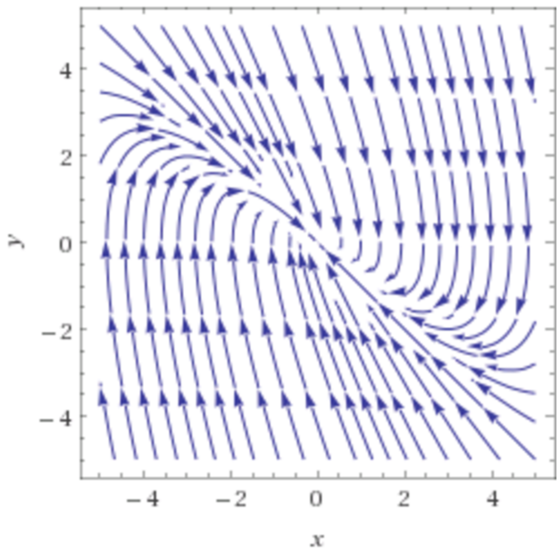
(a) $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\mathbf{x}$

(b) $\mathbf{x}' = \begin{pmatrix} 5 & 0 \\ 2 & -1 \end{pmatrix}\mathbf{x}$

(c) $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 2 & 3 \end{pmatrix}\mathbf{x}$

(d) $\mathbf{x}' = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix}\mathbf{x}$

Solution. (a) First we compute the eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$, so $\lambda_1 = -2, \lambda_2 = -1$. To get an eigenvector for -2, we solve $\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$. So $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = c\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. So we can use $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ as an eigenvector for -2. Similarly, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector for -1. The general solution is
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t}.$$
Plotting the trajectories on the x_1x_2-plane, with arrows indicating the direction as t increases, we get the phase portrait

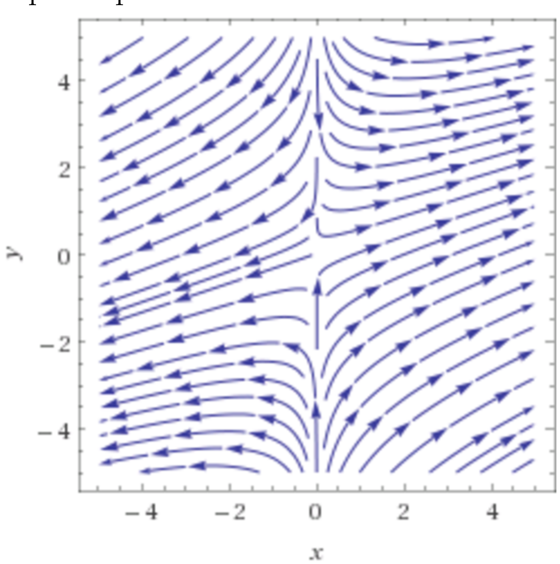


Alternatively, one can plot a direction field using the given system in matrix form, without solving first: at a point **x** the direction vector should be $\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{x}$. The trajectories of the solutions are just the flow lines of the direction field.

(b) Similar to (a), one gets eigenvalues 5 and -1, with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. So the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}.$$

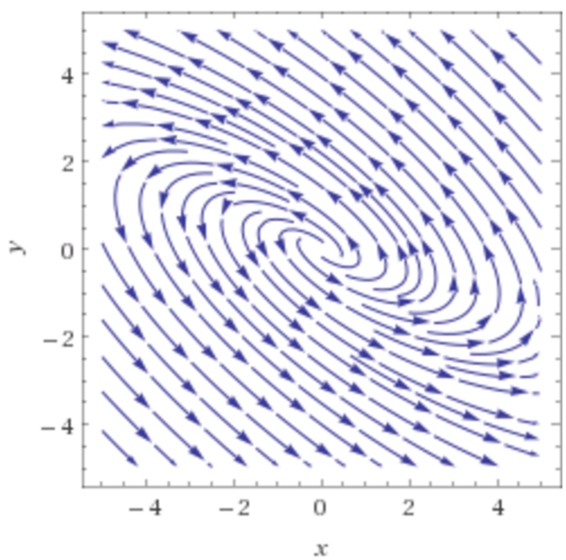
The phase portrait looks like



(c) The eigenvalues are $1+2i$ and $1-2i$, with corresponding eigenvectors $\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1-i \\ 1 \end{pmatrix}$. So the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} -\cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} \cos 2t - \sin 2t \\ \sin 2t \end{pmatrix} e^t.$$

The phase portrait looks like



(d) The only eigenvalue is 5, with corresponding eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (also a repeated eigenvector). To find the generalized eigenvector, we solve

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

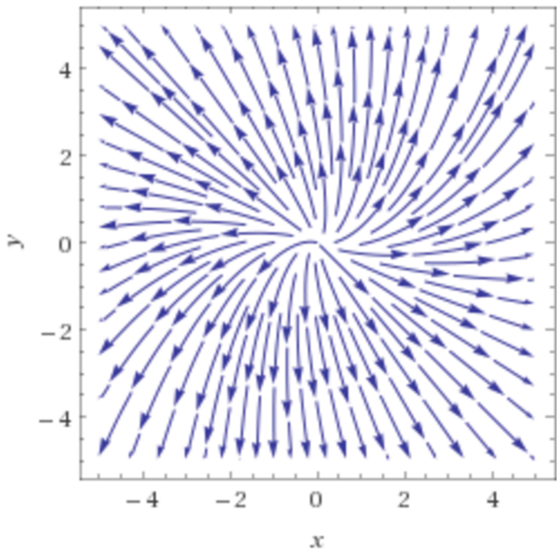
and get

$$\boldsymbol{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

So the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{5t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{5t} \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

The phase portrait looks like



Problem 3. Find a series solution with center $x = 0$ to the differential equation

$$y'' + xy' - 3y = 0.$$

What is the radius of convergence?

Solution. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$. Plugging these series into the equation, we get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} 3 a_n x^n = 0$$

In this new equation, the coefficients for each power of x on the left hand side must add up to zero. So for $n = 0$ we get $2a_2 = 3a_0$, i.e. $a_2 = \frac{3}{2} a_0$. For $n > 0$ we get

$$(n+2)(n+1) a_{n+2} + n a_n - 3 a_n = 0,$$

so

$$a_{n+2} = \frac{(3-n) a_n}{(n+2)(n+1)}$$

In particular, when $n = 3$ we get $a_5 = 0$ from this recursion, so $a_{2n+1} = 0$ for all $n \geq 2$, and when $n = 1$ we get $a_3 = a_1/3$. Observe in fact that the first series equation gives $a_1 = 0$ anyway, so $a_n = 0$ for all odd n. For even indices, we repeat the recursion down to a_0 to get that for $n \geq 1$,

$$a_{2n} = \left(\prod_{k=0}^{n-1} \frac{3-2k}{(2k+2)(2k+1)} \right) a_0.$$

So the general solution is

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} \left(\prod_{k=0}^{n-1} \frac{3-2k}{(2k+2)(2k+1)} \right) x^{2n} \right).$$

Since the given differential equation is of form $y'' + p(x)y' + q(x)y = 0$ and p and q are polynomials, the series solutions have radius ∞.

Problem 4. Given a solution $y_1(x) = e^x$ for the following ODE, find a second independent solution of:

$$(x-1)y'' - xy' + y = 0, \ \ x > 1$$

Solution. Use reduction of order. Suppose $y_2(x) = u(x)e^x$. Calculate $y'(x)$ and $y''(x)$ and plug into the equation. We end up with

$$u'' + \frac{(x-2)}{(x-1)}u' = 0$$

Let $v = u'$; then $v' = u''$. Hence we get a first order ode:

$$v' + \frac{(x-2)}{(x-1)}v = 0$$

Which can be solved by multiplying with an integrating factor $e^{\int \frac{(x-2)}{(x-1)}dx} = e^{x - \ln|x-1|} = \frac{e^x}{x-1}$; or by separation of variables giving $v(x) = c(x-1)e^{-x}$. But $v = u'$.Therefore $u = \int v(x)dx = c(-xe^{-x})$. Thus $y_2 = e^x(cxe^{-x}) = cx, c$ arbitrary.

Problem 5. Find a lower bound on the radius of convergence for series solutions about $x = 0$ of each of the differential equations:
(a) $(x^2 - x - 2)y'' + (x+3)y' - 7y = 0$
(b) $(x^2 - 4x + 5)y'' + y' + x^2y = 0$

Solution. Write the equation as $y'' + p(x)y' + q(x)y = 0$

(a) $p(x) = \frac{(x+3)}{(x^2-x-2)}$ and $q(x) = \frac{-7}{(x^2-x-2)}$. The zeros of the denominator are $x = -1, x = 2$. Lower bound on radius of converegence for solution is given by the minimum distance from the point about which the series is formed, $x = 0$, to the singular points of $p(x), q(x)$. Therefore lower bound on radius of convergence is 1.

(b)By the same reason as above (we get zeros $2 \pm i$) the lower bound on radius of convergence is $\sqrt{5}$.

Problem 6. Use separation of variables to replace the given partial differential equation with a pair of ordinary differential equations:
(a) $xf_{xy} + f = f_{yy}$
(b) $3f_{xx} - xf_y = 0$

Solution. (a) Assume $f(x,y) = X(x)Y(y)$. Then $f_x = X'(x)Y(y)$, $f_{xy} = X'Y'$ and $f_{yy} = XY''$. Plugging these into the given PDE, we get

$$xX'Y' = XY'' - XY$$

Separating for the variables x, y, we get

$$x\frac{X'}{X} = \frac{Y'' - Y}{Y'}$$

For the above to be equal for all x and all y they must be equal to a constant value λ. This gives us two ODEs:

$$xX' - \lambda X = 0$$

and

$$Y'' - \lambda Y' - Y = 0$$

(b) Again assume $f(x,y) = X(x)Y(y)$. Calculate f_{xx}, f_y and plug into the given equation to get

$$\frac{X''}{xX} = \frac{Y'}{3Y} = \lambda$$

Separating into 2 equations we get

$$X'' - \lambda x X = 0$$

and

$$Y' - \lambda 3Y = 0$$

Problem 7. Find two different Fourier series representation of the function $f(x) = 2x$, $0 \leq x \leq 1$. Comment on the convergence of each series.

Solution. One solution is to extend f to an even function, by defining the extension to be $-2x$ on $[-1,0]$, and have period 2. The corresponding Fourier series is a cosine series and has coefficients

$$a_0 = 2\int_0^1 2x\mathrm{d}x = 2$$

and for $n > 0$, using integration by parts,

$$\begin{aligned} a_n &= 2\int_0^1 2x\cos n\pi x\mathrm{d}x = 4\left[\frac{x}{n\pi}\sin n\pi x + \frac{1}{n^2\pi^2}\cos n\pi x\right]_0^1 \\ &= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{8}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

i.e. the series is

$$1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}.$$

Since the extension is continuous everywhere, the series converges to it (and in particular to f) everywhere.

One can also extend to a function of period 2 that is odd near zero, by declaring the extension to be $2x$ on $(-1,0)$. The corresponding Fourier series is a sine series and has coefficients

$$\begin{aligned} b_n &= 2\int_0^1 2x\sin n\pi x\mathrm{d}x = 4\left[-\frac{x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1 \\ &= \frac{-4}{n\pi}(-1)^n, \end{aligned}$$

i.e. the series is

$$-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{n}.$$

At $x = 2k+1$, this series converges to the average of the left and right limits of the extension, i.e. to zero. On the other hand, the extension is always 2 at $2k+1$. So the series converges to the extension for all $x \neq 2k+1$, and in particular converges to f on $[0, 1)$.

Problem 8. Solve the equation below by making a change of variable $u = \ln(\frac{y}{k})$.

$$y' = ry \ln(k/y),\ y(0) = y_0$$

Solution. Using the suggested change of variable $u(t) = \ln(y(t)) - \ln(k) \implies \frac{du}{dt} = \frac{du}{dy}\frac{dy}{dt} = \frac{1}{y}\frac{dy}{dt} \implies \frac{dy}{dt} = y\frac{du}{dt}$. Note that $y(t) = ke^{u(t)}$. Plugging these into the equation we get

$$ke^u u' = -rke^u u(t),\ u(0) = \ln(y_0/k)$$

This is a straightforward equation to solve, which gives after substituting back $u = \ln(\frac{y}{k})$, $y(t) = k\exp[ce^{-rt}]$ where $c = [\ln(y_0/k)]$. (This equation is known as Gompertz equation.)

Problem 9. Find the solution of the initial value problem:

$$2y'' - 3y' + y = 0,\ y(0) = 2, y'(0) = \frac{1}{2}$$

Find the maximum value of the solution and also the point where the solution is zero.

Solution. Characteristic polynomial is $2r^2 - 3r + 1 = 0$. Roots are $r_1 = 1, r_2 = 1/2$ Therefore general solution is $y(x) = c_1 e^t + c_2 e^{t/2}$. Using initial conditions we get $c_1 = -1, c_2 = 3$. Hence general solution is $y(x) = -e^t + 3e^{t/2}$. find $y'(x)$ set it equal to zero to find max/min points. Max occurs at $t = 2\ln(\frac{3}{2})$. At this point $y(x) = \frac{9}{4}$. $y(x) = 0$ when $e^t = 3e^{t/2}$. That is then $\ln(1/3) = -t/2 \implies t = \ln 9$.