Math 23: Differential Equations

Midterm Exam 2

October 2, 2018

NAME: ____________________

SECTION (check one box): Section 1 (A. Gelb 9L) ☐
Section 1 (S. Lin 2) ☐

Instructions:

1. Please turn off or silence cell phones and other electronic devices which may be disruptive.
2. Unless otherwise stated, you must justify your solutions to receive full credit. We will not grade illegible work, or work that is scratched out.
3. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
4. This exam is **closed book**. You may not use notes, or other external resource. You may use calculators. It is a violation of the honor code to give or receive help on this exam. You may of course ask for clarification from the exam proctor on any problem.

Honor statement: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

Signature: ____________________

Problem	Points	Score
1	20	
2	15	
3	25	
4	25	
5	15	
Total	100	

1. *[20 points]* TRUE/FALSE: You must provide a concise justification for your answer. If you claim the statement is false, a counter-example is sufficient.

   (a) Consider the system

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2t & t^2 \\ 3t & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

   If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are both solutions, then so is $3\mathbf{x}^{(1)} + \mathbf{x}^{(2)}$.

   (b) Let $\mathbf{x}' = A\mathbf{x}$, where A a 2×2 real-valued matrix. If $r = 0$ is an eigenvalue of A the system has infinitely many critical points of which $(0, 0)$ is one.

   (c) The particular solution for $y'' + 2y' + 5y = 4e^{-t} \cos 2t$ is of the form $Ae^{-t} \cos 2t + Be^{-t} \sin 2t$.

(d) The critical point $(0,0)$ for the non-linear system of equations

$$\begin{aligned}\frac{dx}{dt} &= (1+x)\sin y \\ \frac{dy}{dt} &= 1 - x - \cos y\end{aligned}$$

is a stable center.

(e) Consider the constant coefficient initial value problem $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ with initial conditions $\mathbf{x}(t_0) = \mathbf{x}^0$. Here $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{x}^0 \in \mathbb{R}^{n\times 1}$. Let $\boldsymbol{\Psi}(t)$ be the fundamental matrix of this system, and $\boldsymbol{\Phi}(t)$ be the fundamental matrix whose column vectors satisfy $\mathbf{x}^{(j)}(t_0) = \mathbf{e}^{(j)}$, where $\mathbf{e}^{(j)}$ is the unit vector with a one in the jth position and zeros everywhere else. (Note that $\boldsymbol{\Phi}(t_0) = I$, the identity matrix.) Then $\boldsymbol{\Phi}(t) = \boldsymbol{\Psi}(t)\boldsymbol{\Psi}^{-1}(t_0)$.

2. [15-points] Match the systems below with the correct phase portraits on the following page.

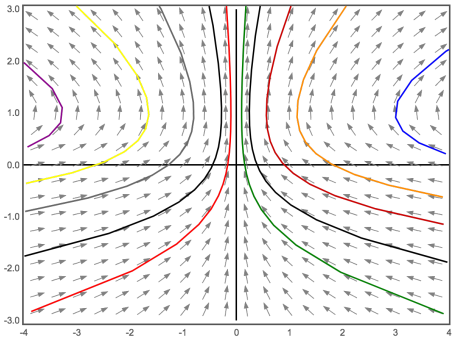
(1) $\frac{dx}{dt} = y^2 - x$; $\frac{dy}{dt} = x^2 + y$.
Figure ______________

(2) $\frac{dx}{dt} = 3x + 8y$; $\frac{dy}{dt} = -4x - 3y$.
Figure ______________

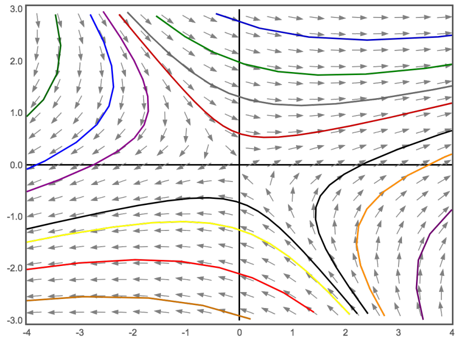
(3) $\frac{dx}{dt} = 2xy - 2x$; $\frac{dy}{dt} = y^2 + 2$.
Figure ______________

(4) $\frac{dx}{dt} = x - 8y$; $\frac{dy}{dt} = 8x + y$.
Figure ______________

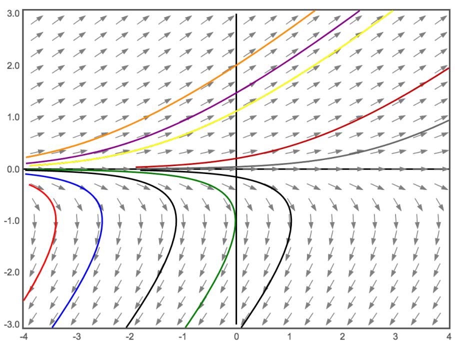
(5) $\frac{dx}{dt} = 2x + 3y$; $\frac{dy}{dt} = x - y$.
Figure ______________



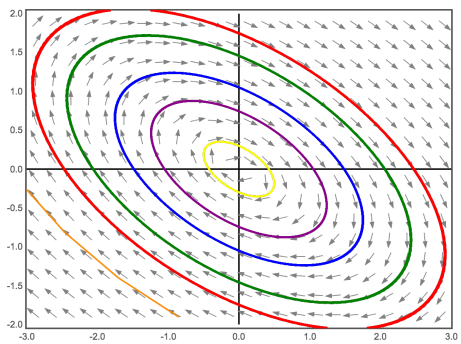
(a)



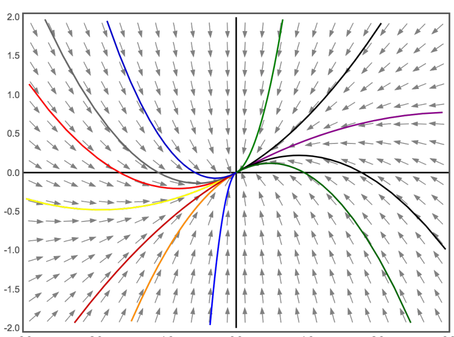
(b)



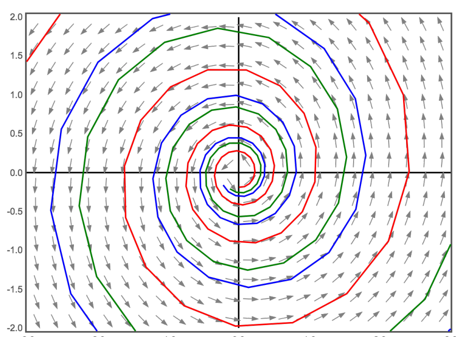
(c)



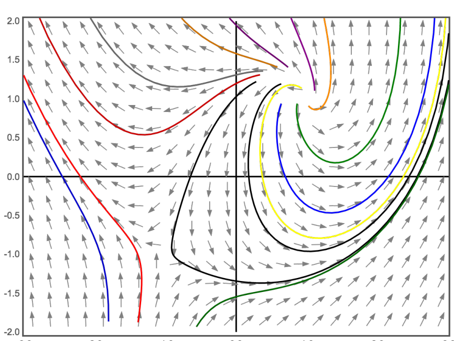
(d)



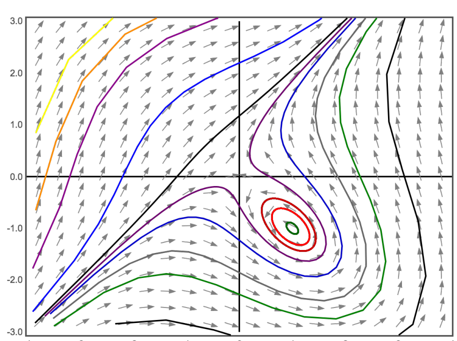
(e)



(f)



(g)



(h)

3. *[25- points]* Consider the system:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$$

(a) Describe the equilibrium point for $\alpha^2 - 4 < 0$. For which values of α it is stable, and for which values is it asymptotically stable? Explain your answer.

(b) For $\alpha^2 - 4 < 0$, choose an α such that the equilibrium point is asymptotically stable. Write the corresponding general solution.

4. *[25- points]* Consider the system:

$$\begin{aligned} x' &= x - xy \\ y' &= y + 2xy \end{aligned}$$

(a) Determine the system's critical points.

(b) Is this system *locally linear* for each critical point? Why or why not?

(c) Write the corresponding linearized system corresponding to each critical point.

(d) Describe the type and the stability of each critical point for the corresponding linearized systems. Be as specific as possible.

(e) What can be said about the type and stability of each critical point for the original non-linear system? Be as specific as possible.

5. [15-points] Consider the second order ODE

$$t^2 y'' + 4ty' + 2y = 2,\ t > 0. \tag{1}$$

(a) Verify that $y_1(t) = t^{-1}$ and $y_2(t) = t^{-2}$ are solutions to the homogeneous equation associated to (1).

(b) Compute the Wronskian of $y_1(t)$ and $y_2(t)$.

(c) Use the variation of parameters to find a particular solution to (1).