Midterm Exam

Math 23

April 22, 2014

Name: Answer Key

Instructions: This exam is closed-book, with no calculators, notes, or books allowed. You may not give or receive any help on the exam, though you may ask the instructor for clarification if necessary. Be sure to show all your work wherever possible.

HONOR STATEMENT: I have neither given nor received any help on this exam, and I attest that all of the answers are my own work.

SIGNATURE: ______________________

Problem	Score	Points
1		15
2		21
3		20
4		22
5		22
Total		100

1. [5 points each]

(a.) Find the general solution of the differential equation: $y' = \cos(t)(y-2)^2$.

Separable : $\frac{dy}{dt} = \cos t\,(y-2)^2$

$$\int \frac{dy}{(y-2)^2} = \int \cos t\, dt \quad , y \neq 2$$

$$-\frac{1}{y-2} = \sin t + c$$

constant soln.

$$y = -\frac{1}{\sin t + c} + 2 \; , y \neq 2 \quad \text{AND} \quad y = 2$$

(b.) For what initial values of t and y are we not guaranteed a unique solution to the following differential equation:

$$y' = \sqrt{1-y^2}.$$

Picard's thm: check when $f = \sqrt{1-y^2}$ is continuous and f_y is continuous.

f is cts when $1-y^2 \geq 0 \Rightarrow -1 \leq y \leq 1$

$f_y = \frac{-y}{\sqrt{1-y^2}}$ must have $1-y^2 > 0 \Rightarrow -1 < y < 1$.

So we are guaranteed unique soln for all t such that $-1 < y < 1$.

(c.) Find the general solution to the differential equation: $y^{(4)} - 16y = 0$.

Char eqn: $r^4 - 16 = 0$.

$$(r^2+4)(r^2-4) = 0$$

$$r = \pm 2i, \pm 2$$

general solution: $y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos 2t + c_4 \sin 2t$

2. [21 points] Let

$$y' = -(3x^2y + 2xy + y^3)/(x^2 + y^2)$$

(a.) Is this equation exact? If not, is there an integrating factor which will make it exact?

(b.) Find the general solution to the equation.

(c.) Find the solution satisfying $y(0) = 3$.

(a) To check exactness, get in right form $M + Ny' = 0$

$$\underbrace{(3x^2y + 2xy + y^3)}_{M} + \underbrace{(x^2+y^2)}_{N}y' = 0$$

is $M_y = N_x$?

$M_y = 3x^2 + 2x + 3y^2$, $N_x = 2x$. Not equal, so not exact.

Check for integrating factor which depends on x only:

$$\frac{M_y - N_x}{N} = \frac{3x^2 + \cancel{2x} + 3y^2 - \cancel{2x}}{x^2+y^2} = 3.$$

so $\mu' = 3\mu \Rightarrow \mu = e^{3x}$.

(b) Find general solution of the form $\varphi(x,y) = c$.

We know $\varphi_x = M$ and $\varphi_y = N$.

where M and N are now $M = e^{3x}(3x^2y + 2xy + y^3)$

and $N = e^{3x}(x^2+y^2)$.

N is easier to integrate with respect to y than M wrt x.

So since $\varphi_y = e^{3x}(x^2+y^2)$,

$$\varphi = e^{3x}x^2y + \frac{e^{3x}y^3}{3} + h(x)$$

Take derivative wrt x $\quad \varphi_x = 3e^{3x}x^2y + 2e^{3x}xy + e^{3x}y^3 + h'(x)$.

Compare w/ M, we find $h'(x) = 0$, so $h(x) = $ Constant.

Therefore, $\varphi(x,y) = e^{3x}x^2y + \frac{e^{3x}y^3}{3} + \text{Constent}$

So the general solution is:

$$\boxed{e^{3x}x^2y + \frac{e^{3x}y^3}{3} = c}$$

(c) If $y(0) = 3$, plug in $x=0$, $y=3$ to solve for c

$e^0 \cdot 0 \cdot 3 + \frac{e^0 \cdot 3^3}{3} = c = 9$, so $\boxed{e^{3x}x^2y + \frac{e^{3x}y^3}{3} = 9}$

3. [20 points] Consider the differential equation

$$t^2y'' + 2ty' - 2y = t^{1/2}.$$

(a.) One solution to the homogeneous problem is $y_1(t) = t$. Using the guess that another solution is of the form $y_2(t) = v(t)y_1(t)$, find the general solution to the homogeneous problem.

(b.) Find the particular solution to this problem.

(a) $y_2(t) = v(t)y_1(t) = v \cdot t$

$y_2' = v't + v$

$y_2'' = v''t + 2v'$

Plug in to the homogeneous problem:

$t^2(v''t + 2v') + 2t(v't + v) - 2vt = 0$

$t^3 v'' + 4t^2 v' = 0$

Let $w = v'$.

$t^3 w' + 4t^2 w = 0$ is separable.

$w' = -\frac{4w}{t}$

$\int \frac{dw}{w} = -\int \frac{4}{t} dt$

$\ln|w| = -4\ln|t|$

$w = t^{-4}$

$v = -\frac{1}{3}t^{-3}$

So $y_2(t) = t \cdot \left(-\frac{1}{3}t^{-3}\right)$

$= -\frac{1}{3}t^{-2}$

So a general solution is

$\boxed{y(t) = C_1 t + C_2 t^{-2}}$

(b) A particular solution $Y_p = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$

where y_1 and y_2 are fundamental solutions.

Let's take $y_1(t) = t$, $y_2(t) = t^{-2}$.

To find v_1, v_2, put DE in correct form:

$$y'' + \frac{2}{t} y' - \frac{2}{t^2} y = t^{-3/2} \Longrightarrow g(t) = t^{-3/2}.$$

$$v_1' = \frac{-y_2 g}{W(y_1,y_2)} \quad \text{and} \quad v_2' = \frac{y_1 g}{W(y_1,y_2)}.$$

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2}.$$

$$v_1' = -\frac{t^{-2} \cdot t^{-3/2}}{-3t^{-2}} = \frac{1}{3} t^{-3/2}, \quad \text{so } v_1 = -\frac{2}{3} t^{-1/2}$$

$$v_2' = \frac{t \cdot t^{-3/2}}{-3t^{-2}} = -\frac{1}{3} t^{3/2}, \quad \text{so } v_2 = -\frac{2}{15} t^{5/2}$$

Therefore $Y_p = -\frac{2}{3} t^{-1/2} \cdot t - \frac{2}{15} t^{5/2} \cdot t^{-2}$

$$\boxed{= -\frac{4}{5} t^{\frac{1}{2}}}$$

4. [22 points] Consider the system:

$$x' = -3x - 2y$$
$$y' = x - y$$

(a) What is the general solution of this equation?

(b) Find the solution with initial conditions $x(0) = 4, y(0) = 0$.

(c) What is the behavior of this solution as t approaches infinity?

(a) $\vec{x}' = \begin{bmatrix} -3 & -2 \\ 1 & -1 \end{bmatrix} \vec{x}$.

Eigenvalues: $\begin{vmatrix} -3-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda) + 2$

$= \lambda^2 + 4\lambda + 5 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16-4(5)}}{2}$

$= -2 \pm i$

Eigenvector for $-2+i$: $\begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(-1-i)\eta_1 = 2\eta_2$ $\qquad \vec{\eta} = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$ $\leftarrow$ one possible eigenvector.

Find Real & Imaginary parts of

$\vec{\eta} e^{\lambda t} = \begin{pmatrix} -1+i \\ 1 \end{pmatrix} e^{-2t}(\cos t + i \sin t) = e^{-2t}\left[\begin{pmatrix} -\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t - \sin t \\ \sin t \end{pmatrix}\right]$

So general solution $\boxed{\vec{x} = c_1 e^{-2t}\begin{pmatrix} -\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} \cos t - \sin t \\ \sin t \end{pmatrix}}$

(b) With initial conditions $x(0)=4, y(0)=0$.

$$\vec{x}(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

$$\vec{x}(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$-c_1 + c_2 = 4$$

$$c_1 = 0 \Rightarrow c_2 = 4$$

So $$\boxed{\vec{x} = 4e^{-2t} \begin{pmatrix} \cos t - \sin t \\ \sin t \end{pmatrix}}$$ ← this should be the same no matter which eigenvector you used.

(c) Since ~~$x<0$~~ Real part <0,

solutions $\longrightarrow \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

as $t \to \infty$.

5. [22 points] Newton's law of cooling says that the rate at which the temperature of a body changes is proportional to the difference between its temperature and the temperature of the surrounding atmosphere. A bottle of water with an initial temperature of $70°F$ cools down to $60°F$ in 1 minute while sitting in a freezer set at $30°F$.

(a.) Assuming the temperature of the water can be described by Newton's Law of Cooling, write down the model that describes this situation.

(b.) What is the constant of proportionality in this example?

(c.) What will the temperature of the bottle of water be after t minutes?

(d.) What happens to temperature of the bottle of water as t approaches infinity, assuming the temperature in the freezer is constant?

(a) y = Temperature of water

$y' = k \cdot (y - y_a)$ where k = constant of proportionality

$y(0) = 70.$ $\quad$ y_a = ambient temperature. $= 30° F$

(b) $y' = k \cdot (y-30).$ Want to solve for k.

$$\frac{dy}{y-30} = k\,dt$$

$$\ln|y-30| = kt + c$$

$$y - 30 = Ce^{kt}$$

$$y = Ce^{kt} + 30$$

$y(0) = C + 30 = 70$, $C = 40$

$$y = 40e^{kt} + 30$$

$y(1) = 60$

$y(1) = 40e^{k} + 30 = 60 \Rightarrow e^{k} = \frac{3}{4} \Rightarrow \boxed{k = \ln\left(\frac{3}{4}\right)}$

(c) Plug in k to formula we've found

$$y = 40e^{\ln(\frac{3}{4})t} + 30$$

$$\boxed{y = 40\left(\frac{3}{4}\right)^t + 30.}$$

(d) as $t \to \infty$,

since $\frac{3}{4} < 1$, $\left(\frac{3}{4}\right)^t \to 0$ as $t \to \infty$.

$$\lim_{t\to\infty} 40\left(\frac{3}{4}\right)^t + 30 = 0 + 30 = \boxed{30}$$

$y \to 30$ as $t \to \infty$