# MATH 23: DIFFERENTIAL EQUATIONS
# WINTER 2017
# PRACTICE MIDTERM EXAM PROBLEMS

**Problem 1**. (a) Find the general solution of the differential equation

$$2y'' + 3y' + y = \sin 2t$$

(b) What is the behavior of the solution as $t \to \infty$?

*Solution.* The characteristic equation for the corresponding homogeneous equation is $2r^2 + 3r + 1 = 0$, with roots $r_1 = -1/2$, $r_2 = -1$. So the general solution to the homogeneous equation is

$$y = C_1 e^{-t/2} + C_2 e^{-t}.$$

We guess $Y = A \sin 2t + B \cos 2t$ as a particular solution to the nonhomogeneous equation. Plugging $Y$ into the nonhomogeneous equation, we get

$$(-7A - 6B) \sin 2t + (6A - 7B) \cos 2t = \sin 2t$$

Solving the system

$$\begin{aligned} -7A - 6B &= 1 \\ 6A - 7B &= 0 \end{aligned}$$

we get $A = -7/85$ and $B = -6/85$, so the general solution to the equation is

$$y = C_1 e^{-t/2} + C_2 e^{-t} - \frac{7}{85} \sin 2t - \frac{6}{85} \cos 2t.$$

As $t \to \infty$, for any choice of $C_1$ and $C_2$ the first two summands of $y$ approach zero, and the sum of the other two summands oscillates with a constant amplitude, so $y$ oscillates without approacing a limit as $t \to \infty$.

**Problem 2**. (a) Find the general solution of the differential equation

$$2y'' - 3y' - y = t^2$$

(b) What is the behavior of the solution as $t \to \infty$?

*Solution.* The solutions to the characteristic equation are $r_1 = \frac{3}{4} + \frac{\sqrt{17}}{4}$ (positive) and $r_2 = \frac{3}{4} - \frac{\sqrt{17}}{4}$ (negative), so the general solution to the corresponding homogeneous equation is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$. The right hand side of the equation is a polynomial of degree two, and there are no polynomial solutions to the homogeneous equation (so we don't need to introduce a factor of $t^s$), so we try $Y = At^2 + Bt + C$ as a particular solution. Plugging $Y$ into the equation, we get

$$4A - 6At - 3B - At^2 - Bt - C = t^2,$$

so $A = -1$, $B = 6$, $C = -4$. Thus, the general solution is

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} - t^2 + 6t - 4.$$

As $t \to 0$, the second summand tends to 0, and the polynomial part to $-\infty$. When the constant $C_1$ is chosen positive, the first term approaches $\infty$ and dominates the polynomial summand, so $y \to \infty$. When $C_1 \leq 0$, the solution approaches $-\infty$.

**Problem 3**. What is the integrating factor $\mu(t)$ used to solve the first order linear equation:

$$2t^2 y' - 6ty = e^{-t}$$

*Solution.* We first need to arrange that the coefficient of $y'$ is 1. So we divide by $2t^2$ and rewrite the equation as

$$y' - \frac{3}{t} y = \frac{e^{-t}}{2t^2}.$$

The integrating factor then is

$$\mu(t) = e^{\int \frac{-3}{t} \mathrm{d}t} = e^{-3 \ln t} = t^{-3}.$$

**Problem 4**. Solve the initial value problem

$$y''' - 3y'' + 4 = 0; \qquad y(0) = 1, y'(0) = 0, y''(0) = -1$$

*Solution.* Use reduction of order. Set $u = y''$ and solve $u' - 3u = -4$.

**Problem 4'**. Solve the initial value problem

$$y''' - 3y'' + 4y = 0; \qquad y(0) = 1, y'(0) = 0, y''(0) = -1$$

*Solution.* The characteristic equation factors as $(r+1)(r-2)^2 = 0$, so the general solution is $C_1 e^{-t} + C_2 e^{2t} + C_3 t e^{2t}$.

**Problem 5**. Determine the *form* of the particular solution when the method of undetermined coefficients is used to solve

$$y'' - 4y = e^{-2t} + 2t^2 - 1$$

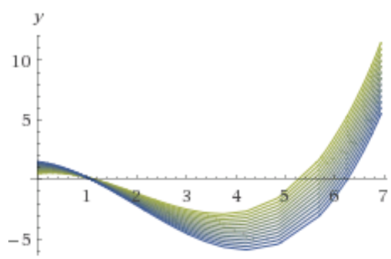
Do NOT solve for the coefficients.

*Solution.* Since the right hand side is a sum of a polynomial and an exponential function, we know that we can find a particular solution of the same form, except that we may need to multiply each summand by $t$ or $t^2$ if the characteristic equation for the corresponding homogeneous equation has a repeated root. Since the characteristic equation is $r^2 - 4 = 0$, with roots $\pm 2$, we see that need to multiply by a power of $t$. Thus, the form of the general solution is

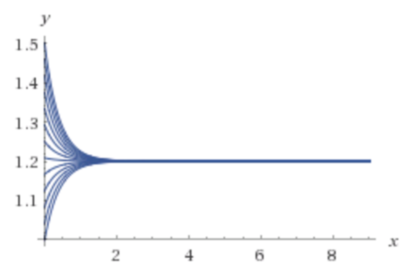
$$Y = Ate^{-2t} + Bt^2 + Ct + D.$$

**Problem 5**. Each differential equation matches a graph of a general solution. Give the letter of the correct graph:

| | |
|---|---|
| (a) $2y' + 5y = 6$ | $B$ |
| (b) $y'' - t = -2$ | $A$ |
| (c) $y' = \sqrt{y}$ | $C$ |



A



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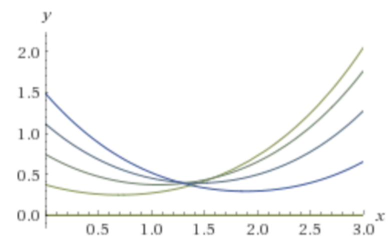

C

**Problem 7**. Show that every solution of the equation $y' = x^3(y^2+1)$ has at most one minimum point.

*Solution.* If $y$ has a minimum at some $x_0$, then it must be that $y'(x_0) = 0$, i.e. $x_0^3(y(x_0)^2+1) = 0$. This can only happen for $x_0 = 0$, so $y$ either attains a minimum at 0 or nowhere.

**Problem 8**. Solve the differential equation

$$y''' = ty''$$

*Solution.* Using reduction of order, we let $u = y''$ and solve $u' = tu$, which is a first order linear equation. We get $u = Ce^{t^2/2}$, so $y' = \int_{t_0}^{t} Ce^{s^2/2}\mathrm{d}s + D$, so

$$y = \int_{t_0}^{t} y'(s)\mathrm{d}s + E = \int_{t_0}^{t}\left(\int_{t_0}^{s} Ce^{q^2/2}\mathrm{d}q + D\right)\mathrm{d}s + E.$$

In this case, $u$ is not very nice to integrate, so you do not need to simplify your answer further.