Sketchy Answers to Selected Exercises, Assignment 1

Math 24 Spring 2006

1.2

7. Equality for functions is based on their output, so you can show $f = g$ and $f + g = h$ by showing the equalities hold for inputs 0 and 1.

8. Distribute the additions one at a time (in either order); rearrange by commutivity if necessary.

9. Do essentially exactly like the corresponding results for fields.

13. No, this is not a vector space: the identity is $(0, 1)$ but any vector of the form $(a, 0)$ has no inverse. Alternately, the proposed vector space violates the property $0x = \mathit{0}$, which follows from fulfilling the vector axioms, for every x not of the form $(a, 1)$. There are probably also other objections.

14, 15. Yes for $\mathbb{C}^n$ over $\mathbb{R}$; no for $\mathbb{R}^n$ over $\mathbb{C}$. The latter is all i's fault, making the proposed vector space not closed under scalar multiplication.

17. No, this is not a vector space, because there is no scalar which acts as a multiplicative identity.

1.3

4. Do componentwise, working from a_{ij}^{tt} to a_{ji}^{t} to a_{ij} (where the t's in the components mark which matrix they came from).

5. Again, work componentwise to show $b_{ij} = b_{ji}$ for b the elements from $A + A^t$.

8. a, c, d are subspaces. b and e fail to have a zero vector. f is not closed under addition; vectors (a, b, c) and $(-a, b, c)$ add to a vector outside W_6 $(a \neq 0)$.

9. The first two intersections are $\{\mathit{0}\}$; the third gives $\{(a, b, c) : a = -11c, b = -3c\}$, which may be shown a subspace by the same method as 8(a).

14. Zero and scalar multiple closure are straightforward. For addition, note that if $(f + g)(x) \neq 0$, at least one of f, g must be nonzero at x, and the total number of points at which that happens cannot exceed the sum of the number of points where f and g are individually nonzero.

20. Proceed stagewise, first asserting all the terms $a_i w_i$ are in W, and then adding them one at a time.

23. Write out the sums and products using the fact that every $v \in W_1 + W_2$ is $w_1 + w_2$ for some $w_i \in W_i$. Rearrange the pieces and use the fact that the W_i are subspaces.

25. The fact that W_i are subspaces and intersect to $\{\mathit{0}\}$ is clear; the fact that they sum to $P(F)$ is a matter of writing down the generic form of an element of $P(F)$ and taking it apart

into two pieces (can say "without loss of generality let the degree be even" for cleaner use of variables).

1.4

11. Span is linear combinations, but with only one element that reduces to scalar multiples. The span of a single element of $\mathbb{R}^3$ is a line.

13. For the first part, use the fact that all the vectors we are taking linear combinations of to get $\text{span}(S_1)$ are also elements of S_2. For the second, note that by the first part $\text{span}(S_2)$ is at least V, but since the elements of S_2 are members of V and V is closed under linear combinations it cannot be more than V.

15. Show that if v is in $\text{span}(S_1 \cap S_2)$, then it is in both $\text{span}(S_1)$ and $\text{span}(S_2)$. The cases are symmetric so you can do just one out in full. If $v \in \text{span}(S_1 \cap S_2)$, then it is a linear combination of vectors in $S_1 \cap S_2$ and in particular S_1. Hence it is in $\text{span}(S_1)$.

1.5

3. Add the first three and subtract off the last two, or vice-versa.

9. Suppose $au + bv = \mathit{0}$ with b nonzero. Then $\frac{a}{-b}u = v$, and conversely. The case is symmetric if $a \neq 0$ (cannot assume both are nonzero).

11. Since the multiples of vectors over $\mathbb{Z}_2$ are either zero or the original vector, the elements of $\text{span}(S)$ are sums of subsets of S (including the subset of size zero, to account for the zero multiple of any element). Each of these is distinct because S is linearly independent, and so we have the same number of sums as S has subsets; that is, 2^n.

[If this is unfamiliar to you, the rule is that a set of n elements has 2^n subsets counting $\varnothing$ and the set itself; to see this, consider the choice of elements for a subset as n independent yes/no questions. Such a series of questions has 2^n possible series of answers.]

16. The downward direction is clear from previous results. For the upward direction, suppose S is linearly dependent. Then for some *finite* set of vectors u_i from S, we have $a_1u_1 + \ldots a_nu_n = \mathit{0}$, a_i not all zero. However, the set $\{u_1, \ldots, u_n\}$ is a finite linearly dependent subset of S, so if no such subset exists, S must be linearly independent as well.