Assignment 8 - selected answers (3.2–3.3)

3.2

3. Clearly $A = 0$ implies A's rank is 0. For the converse, if $A \neq 0$ it must have at least one nonzero column and hence the columns must span a vector space of dimension at least 1.

4. (b) rank is 2, matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

6. (a) T^{-1} can also be expressed as $T(f(x)) = -(f(x) + 2f'(x) + f''(x))$.
(b) $T(1) = 0$, so noninvertible.
(d) $T^{-1}(b_1 + b_2x + b_3x^2) = (b_3, \frac{1}{2}(b_1 - b_2), \frac{1}{2}(b_1 + b_2) - b_3)$.
(f) Since $\mathrm{tr}(A)$ and $\mathrm{tr}(A^t)$ are equal, T is not onto and hence noninvertible.

8. Let v_i be the columns of A and $w_i = cv_i$ be the columns of cA. If $x = a_1v_1 + \ldots + a_nv_n$, then because $c \neq 0$, $x = \frac{1}{c}(a_1w_1 + \ldots + a_nw_n)$.

14. (a) Show $R(T + U) \subseteq R(T) + R(U)$:
Let $x \in R(T + U)$. Then there is a y with $(T + U)(y) = x$; i.e., $T(y) + U(y) = x$ and so $x \in R(T) + R(U)$.
(b) Show the rank of the sum is bounded by the sum of the ranks:
Certainly if β is a basis for $R(T)$ and γ a basis for $R(U)$, the set $\{b + 0, 0 + g : b \in \beta, g \in \gamma\}$ spans $R(T) + R(U)$, and so the dimension of $R(T) + R(U)$ is bounded by $\mathrm{rank}(T)$ plus $\mathrm{rank}(U)$. Then since $R(T + U)$ is a subspace by part (a) the result follows.
(c) ......

3.3

2. (b) $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$

3. (a) Other vectors besides $(5, 0)$ would work (e.g., $(2, 1)$).
(b) $\left\{ \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} + a \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$

4. (a) $A^{-1} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$, solution $\begin{pmatrix} -11 \\ 5 \end{pmatrix}$

6. $(11/2, -9/2, 0)$ could be replaced, e.g. by $(0, 1, -11)$