Mathematical Induction

Notes and worksheet for Math 24, Spring 2006

The Replacement Theorem (Theorem 1.10 in your text) is proved by induction. It is possible that at this point you've had only a passing acquaintanceship with induction or even none at all.

The basic premise of induction is that if you can start, and once you start you know how to keep going, then you will get all the way to the end. If I can get on the ladder, and I know how to get from one rung to the next, I can get to the top of the ladder.

Principle of Mathematical Induction, basic form:

If S is a subset of the positive integers such that $1 \in S$ and $n \in S$ implies $n+1 \in S$ for all n, then S contains all of the positive integers. [We may need the beginning to be 0 or another value depending on context.]

Use of mathematical induction. In general you want to use induction to show that some property holds no matter what integer you feed it, or no matter what size finite set you are dealing with. The proofs always have a *base case*, the case of 1 (or wherever you're actually starting). Then they have the *inductive step*, the point where you assume the property holds for some unspecified n and then show it holds for $n+1$.

An example:

Prove that for every positive integer n, the equation

$$1 + 3 + 5 + \ldots + (2n-1) = n^2$$

holds.

Proof:

Base case: For $n = 1$, the equation is $1 = 1^2$, which is true.

Inductive step: Assume that $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for some $n \geq 1$. To show that it holds for $n+1$, add $2(n+1)-1$ to each side, in the simplified form $2n+1$:

$$1 + 3 + 5 + \ldots + (2n-1) + (2n+1) = n^2 + 2n + 1 = (n+1)^2.$$

Since the equation above is that of the theorem, for $n+1$, by induction the equation holds for all n.

Another example:

Prove that for $n > 2$, the sum of angle measures of the interior angles of a convex polygon of n vertices is $(n-2) \cdot 180°$.

Proof:

Base case: Here we begin at $n = 3$ instead of 0 or 1. For $n = 3$, the polygon in question is a triangle, and it has interior angles which sum to $180° = (3-2) \cdot 180°$.

Inductive step: Suppose the theorem holds for $n \geq 3$ and consider a convex polygon with $n+1$ vertices. Since $n+1 \geq 4$, if we take one vertex x there is another vertex y such that there is one vertex between x and y in one direction, and $n-2$ in the other direction. Join x and y by a new edge, dividing our original polygon into two polygons. The new polygons' interior angles summed together make up the sum of the original polygon's interior angles. One of the new polygons is a triangle and the other a polygon of n vertices (all but the one isolated between x and y). The triangle has interior angle sum $180°$, and by the inductive hypothesis the other polygon has interior angle sum $(n-2)\cdot 180°$. Summing these we get $(n+1-2)\cdot 180°$, and the theorem is proved.

As you get more comfortable with induction, you can write it in a more natural way, without segmenting off the base case and inductive step portions of the argument.

Induction proofs to try:

(1) For every positive integer n,
$$1+4+7+\ldots+(3n-2) = \frac{1}{2}n(3n-1).$$

(2) For every positive integer n,
$$2^1+2^2+\ldots+2^n = 2^{n+1}-2.$$

(3) For every positive integer n, $\dfrac{n^3}{3}+\dfrac{n^5}{5}+\dfrac{7n}{15}$ is an integer.

(4) For every positive integer n, 4^n-1 is divisible by 3.

(5) The sequence $a_0, a_1, a_2, \ldots$ defined by $a_0 = 0$, $a_{n+1} = \frac{a_n+1}{2}$ is bounded above by 1.

(6) Recall that for a binary operation $*$ on a set A associativity is defined as "for any x, y, z, $(x*y)*z = x*(y*z)$." Use induction to prove that for any collection of n elements from A being put together with $*$, $n \geq 3$, any grouping of the elements which preserves order will give the same result.