Math 24 Spring 2006 Final Exam
Review Guide

A. Some items to know:

(1) The definition and how to check it for the following:
   (a) vector space
   (b) subspace
   (c) linearly independent set
   (d) linear transformation

(2) Standard examples of the following:
   (a) vector spaces, with their dimensions and standard bases
   (b) linear transformations (including I_V and T_0), with their null space and range

(3) Cancellation and that multiplication by zero gives zero in a vector space.

(4) That the intersection of two subspaces is a subspace but the union almost never is.

(5) Linear combinations, linear dependence and independence, generating/spanning sets, bases, trivial and nontrivial representations of $\mathit{0}$, dimension.

(6) That any linearly independent set may be built up to a basis and any spanning set pared down into one; given a linearly independent set L of size n any spanning set may have n of its elements swapped out for L and the result will still span (though not just any n elements may be swapped out). The consequence that any linearly independent set or spanning set of the correct size is a basis.

(7) That linear transformations are determined entirely by what they do to a basis (and that for anything you want to do to it, there is a transformation), and that the basis images can be used to make a matrix representation of T.

(8) Null space (kernel) and range (image); that $N(T)$ is a subspace of the domain and $R(T)$ of the codomain. Nullity and rank and their relationship to the dimension of the domain.

(9) Characterizations of one-to-one and onto in terms of null space and rank (remember the limitation that to relate one-to-one and onto the domain and codomain must have equal dimension).

(10) Ordered bases and the distinction between vectors and coordinate vectors.

(11) How to add and scalar-multiply linear transformations, and that the set of all linear maps between two fixed vector spaces are themselves a vector space. How to multiply matrices. That matrix multiplication and linear transformation composition are tightly related.

(12) Isomorphism and the characterization of vector spaces up to isomorphism by their dimension. Invertibility for transformations and matrices and their relationship.

(13) The left-multiplication transformation and the fact that it is the inverse operation to matrix representation of transformations in terms of the standard basis (and hence that $\mathcal{L}(V, W)$ and $M_{\dim W \times \dim V}(F)$ are isomorphic).

(14) Change of coordinates matrices and using them to simplify the finding of linear transformation matrices. Definition of similarity.

(15) Elementary row and column operations and matrices; their relationship to each other; the matrices' relationship to their inverses; that every invertible matrix is the product of elementary matrices.

(16) Rank for a matrix; that multiplication by invertible matrices preserves rank; the alternate characterizations of rank as the maximum number of linearly independent rows or columns and the attendant restriction on the maximum rank of a matrix.
(17) That rank cannot increase when multiplying matrices or composing linear transformations.
(18) That by using row and column operations, any matrix may be transformed into one which has an identity matrix in the upper left-hand corner and zeros elsewhere (and hence rank is preserved between a matrix and its transpose). Using row operations only, the matrix can be transformed into reduced row echelon form.
(19) The augmented matrix (for A or $Ax = b$) and how to use row operations to reduce it to find the rank (and inverse, if it exists) of A or the solution to $Ax = b$. You will not be required to use Gaussian elimination (create leading 1s left to right with 0s below them and then work right to left to put 0s above the leading 1s) but as it is provably the most efficient way to row-reduce, you are encouraged to adopt it.
(20) That reduced row echelon form is unique, and that the linear dependence relationships between its columns are the same as in the original matrix, and therefore it may be used to find bases out of spanning sets.
(21) The terms homogeneous, nonhomogeneous, consistent, inconsistent, and equivalent (applied to systems of linear equations in all cases). The characterization of the solution set of a homogeneous system as a null space.
(22) The relationship between $Ax = b$ and $Ax = \mathit{0}$; the fact that an invertible A gives a system with exactly one solution, $A^{-1}b$. The breakdown of a general solution for $Ax = b$ into a single solution for $Ax = b$ and a basis for the solution set of $Ax = \mathit{0}$.
(23) Definition of determinant and how to find it by cofactor expansion; relationship between determinant and invertibility; that determinant is unchanged by taking transpose or finding a similar matrix.
(24) What it means for a matrix to be diagonalizable and the relationship to change of coordinate matrices.
(25) The three eigen-words and how they relate to each other; how to find them from a given linear transformation or matrix. Remember eigenvalues may be zero but eigenvectors are nonzero. Characteristic polynomial and how to find it.
(26) What can go wrong when finding eigenvalues and eigenspaces – i.e., the characterization of when a matrix is diagonalizable. Definition of splitting and multiplicity.

B. Some notes on proofs.

I will ask you at least one old proof on the exam (from the book or from homework), and there will be something new to prove on the exam. Memorizing is not as good a use of your time as trying to understand what's going on. Results which we continued to use through the term, with relatively short proofs, are the best candidates for the old proof.

Common techniques:

(1) Addition of 0 (or $\mathit{0}$) or multiplication by 1 (or I) in some form.
(2) Using the (high-level) definition of a vector operation or function on vector spaces to put things in terms of the (low-level) entries of a vector where you can use properties of, say, ordinary arithmetic.

(3) Noting that a certain constant c is nonzero and using that fact to solve a linear combination for the vector which has c as a coefficient.
(4) Building a particular basis for a vector space by starting with a given linearly independent set (such as the basis of a subspace).
(5) Using the fact that a transformation is linear to transfer properties between the domain and codomain.
(6) Constructing a linear transformation that maps a basis to a set you specify and proving it has desired properties.
(7) Shifting between linear transformations and their matrix representations.
(8) For rank or consistency, simplifying the matrix in a legal way to get a matrix for which the answer is clear.
(9) Use of some basic results:
   (a) Big-enough linearly independent sets and small-enough spanning sets are bases; the size of a spanning set limits the maximum dimension of a vector space.
   (b) The image of a basis spans the range of a linear transformation.
   (c) The nullity and rank of a transformation sum to the dimension of its domain; nullity of zero is equivalent to being one-to-one.

It is often best to work directly from the definition (we do it all the time when checking a transformation is linear or a subset of a vector space is a subspace). In fact, many proofs are nearly as short proved directly as they are using any of the results listed in #9 above. A note from quiz 2: think carefully in a problem whether you want to work with matrices as a whole or element-wise. Sometimes using, say, the definition of matrix multiplication is necessary, but if you can, it is generally nicer to assert things about the matrix as a whole (for example, if you're working with invertibility, look at what can be done with $A^{-1}A = I$ rather than the linear independence of the rows or columns of A).

C. Some summations.

I. For an $n \times n$ matrix A over a field F, the following are equivalent:
(1) A is invertible.
(2) $\text{rank}(A) = n$.
(3) $\text{rank}(A^t) = n$.
(4) The columns of A are all linearly independent.
(5) The rows of A are all linearly independent.
(6) For every $b \in F^n$, there is exactly one $a \in F^n$ such that $Aa = b$.
(7) The determinant of A is nonzero.
(8) All of the eigenvalues of A are nonzero.

Some would say that determining when a matrix/linear operator is invertible is the point of linear algebra, and so the list above is the "key theorem" of linear algebra.

II. A flowchart of diagonalization; dimension is n and matrix A (which is $[T]_\beta$ if starting with a linear transformation T). Theory on the left and practice on the right.

Remember that *everything* in the process comes from the equation $Ax = \lambda x$.

0. Put the linear transformation into matrix form.

Eigenvalues are λ such that $\det(A - \lambda I_n) = 0$. That is, they are values which make that difference of matrices noninvertible, so it has null space more than just zero.

1. Find the characteristic polynomial and its roots. Does it have n roots (including repeats)?
   - *No* $\longrightarrow$ A is not diagonalizable, but any roots it does have are eigenvalues and step 2 can be used to find their eigenvectors.
   - *Yes* $\longrightarrow$ A may be diagonalizable; find all roots and proceed to step 2.

Eigenvectors are x such that $(A - \lambda I_n)x = \mathit{0}$. Distinct eigenvalues have linearly independent nonzero eigenvalues; that is, their eigenspaces intersect to $\{\mathit{0}\}$. Each eigenspace has dimension at most the multiplicity of the eigenvalue, so each must be maximal to have a basis of eigenvectors.

2. For each eigenvalue λ found in Step 1, find the null space of $L_{A-\lambda I_n}$. Is its dimension the same as the multiplicity of λ?
   - *No* (for any) $\longrightarrow$ A is not diagonalizable, though all elements of the null space are eigenvectors.
   - *Yes* (for all) $\longrightarrow$ A is diagonalizable; the diagonal elements are the eigenvalues, with order corresponding to the ordered basis of eigenvectors.