Math 24

Winter 2014

Thursday, January 23

(1.) TRUE or FALSE? In these exercises, $V$ and $W$ are finite-dimensional vector spaces over the same field $F$, and $T$ is a function from $V$ to $W$.

a. If $T$ is linear, then $T$ preserves sums and scalar products. (T)

   This is the definition of linear.

b. If $T(x + y) = T(x) + T(y)$, then $T$ is linear. (F)

   We must also have $T(ax) = aT(x)$.

c. $T$ is one-to-one if and only if the only vector $x$ such that $T(x) = 0$ is $x = 0$. (F)

   This is true if $T$ is linear.

d. If $T$ is linear, then $T(0_V) = 0_W$. (T)

e. If $T$ is linear, then $nullity(T) + rank(T) = dim(W)$. (F)

   $nullity(T) + rank(T) = dim(V)$.

f. If $T$ is linear, then $T$ carries linearly independent subsets of $V$ onto linearly independent subsets of $W$. (F)

   For example, the zero transformation $T(x) = 0$, carries linearly independent sets onto $\{0\}$. Any linear $T$ does, however, carry linearly dependent subsets of $V$ onto linearly dependent subsets of $W$.

g. If $T : V \to W$ and $U : V \to W$ are both linear and agree on a basis for $V$ (meaning that if $x$ is in the basis, $T(x) = U(x)$), then $T = U$. (T)

h. Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. (F)

   For example, if $x_2 = 2x_1$ but $y_2 \neq 2y_1$, there is no such $T$. This is true, however, if $\{x_1, x_2\}$ is linearly independent.

i. Recall that we can consider $\mathbb{R}$ to be a vector space over itself. Any function $T : \mathbb{R} \to \mathbb{R}$ of the form $T(x) = mx + b$, where $m$ and $b$ are constants in $\mathbb{R}$, is linear. (F)

   If $b \neq 0$, then $T(0) \neq 0$, so $T$ cannot be linear. This is an *affine* function, the sum of a linear function and a constant function. If $b = 0$, it is linear.

j. The words "range," "image," and "codomain" all mean the same thing. (F)

   Range and image denote $R(T)$; codomain denotes $W$.

k. If $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ is linear and $N(T)$ is the subspace of diagonal matrices in $M_{2\times 2}(\mathbb{R})$, then $T$ is not onto. (T)

We have $n(T) = 2$ and $dim(domain(T)) = 4$, so by the dimension theorem, $r(T) = 2$. Since the codomain has dimension 3, $T$ s not onto.

(2.) Explain why we know that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ is not linear.

Just for fun, we'll use a different argument for each example.

a. $T(a_1, a_2) = (1, a_2)$.

$T$ does not send zero to zero: $T(0, 0) \neq (0, 0)$.

b. $T(a_1, a_2) = (a_1, (a_1)^2)$.

$T$ does not preserve scalar products: $2(T(1, 0)) \neq T(2(1, 0))$.

c. $T(a_1, a_2) = (\sin(a_1), 0)$.

The null space of $T$ is not a subspace of $\mathbb{R}^2$. (It includes $(\pi, 0)$ but not $\frac{1}{2}(\pi, 0)$.)

d. $T(a_1, a_2) = (|a_1|, a_2)$.

The range of $T$ is not a subspace of $\mathbb{R}^2$. (It includes $(1, 1)$ but not $-(1, 1)$.)

e. $T(a_1, a_2) = (a_1 + 1, a_2)$.

$T$ does not preserve sums: $T((0, 0) + (0, 0)) \neq T(0, 0) + T(0, 0)$.

(3.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x - y, 2y - z, 4x - z)$.

a. Find a basis for the null space of $T$.

We need to find a basis for the solution space for the system of linear equations

$$2x - y = 0$$

$$2y - z = 0$$

$$4x - z = 0.$$

Gaussian elimination converts this system to

$$\boxed{x} - \frac{1}{4}z = 0$$

$$\boxed{y} - \frac{1}{2}z = 0$$

$$0 = 0.$$

We use the first two equations to solve for $x$ and $y$, and set $z$ equal to a parameter, $s$:

$$(x, y, z) = \left(\frac{1}{4}s,\ \frac{1}{2}s,\ s\right) = s\left(\frac{1}{4},\ \frac{1}{2},\ 1\right).$$

The null space consists of all vectors of this form, and a basis is

$$\left\{\left(\frac{1}{4},\ \frac{1}{2},\ 1\right)\right\}.$$

b. Find a basis for the range of $T$.

Since the domain of $T$ is spanned by the set $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$, the range of $T$ is spanned by the set

$$\{T(1,0,0),\ T(0,1,0),\ T(0,0,1)\} = \{(2,0,4),\ (-1,2,0),\ (0,-1,-1)\}.$$

Since this set spans the range, we can reduce it to a basis, by considering its elements one by one and eliminating any that is a linear combination of the earlier ones. This gives us the basis

$$\{(2,0,4),\ (-1,2,0)\}.$$

c. Find the nullity and rank of $T$. Verify the dimension theorem (in the case of $T$).

The dimension theorem tells us that

$$n(T) + r(T) = dim(domain(T)).$$

The nullity of $T$ is the dimension of the null space, in this case $n(T) = 1$, the rank of $T$ is the dimension of the range, in this case $r(T) = 2$, and in this case the domain of $T$ is $\mathbb{R}^3$, so $dim(domain(T)) = 3$. It is true that

$$1 + 2 = 3,$$

which verifies the dimension theorem in this case.

d. Is $T$ one-to-one? How can you tell from the nullity and/or rank of $T$?

$T$ is not one-to-one. We can tell because the nullity of $T$ is not zero.

e. Is $T$ onto? How can you tell from the nullity and/or rank of $T$?

$T$ is not onto. We can tell because the rank of $T$ does not equal the dimension of the codomain.