Math 24
Winter 2014
Friday, January 24

1. Assume V and W are finite-dimensional vector spaces with ordered bases β and γ respectively, and T and U are linear transformations from V to W.

   TRUE or FALSE?

   (a) For any scalar a, $aT + U$ is a linear transformation from V to W. (T)
   (b) $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies that $T = U$. (T)
   (c) If $m = dim(V)$ and $n = dim(W)$ then $[T]_\beta^\gamma$ is an $m \times n$ matrix. (F) (It's $n \times m$.)
   (d) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$. (T)
   (e) $\mathcal{L}(V, W) = \mathcal{L}(W, V)$. (F) (They do have the same dimension, mn.)
   (f) If $m = dim(V)$ then the function $f : V \to F^m$ defined by $f(v) = [v]_\beta$ is a linear transformation from V to F^m. (T)
   (g) Every element of $M_{3\times3}(\mathbb{R})$ is the matrix of some linear transformation from $\mathbb{R}^3$ (with the standard ordered basis) to $P_2(\mathbb{R})$ (with the standard ordered basis). (T)
   (h) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that $[T]_\beta^\beta = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$, where β is the standard ordered basis for $\mathbb{R}^2$, then $T(1, -1) = (1, 4)$. (F) (It's $(-2, 1)$.)

Here is the computation from that last item. The standard ordered basis is $\beta = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$.

The columns of the matrix, therefore, are the coordinates of the images the vectors in β, that is, the coordinates of $T(e_1)$ and $T(e_2)$.

Because the ordered basis of the codomain is β, when we say "coordinates" in the previous sentence, we mean "β coordinates." That is, we see from $[T]_\beta^\beta$ that

$$[T(e_1)]_\beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad [T(e_2)]_\beta = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Because the designated codomain basis vectors are e_1 and e_2, this tells us that

$$T(e_1) = 2e_1 + 1e_2 = (2, 1) \quad \text{and} \quad T(e_2) = 4e_1 + 0e_2 = (4, 0).$$

Therefore, because T is linear, we can write

$$T(1, -1) = T((1, 0) - (0, 1)) = T(e_1 - e_2) = T(e_1) - T(e_2) = (2, 1) - (4, 0) = (-2, 1).$$

2. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined as follows:

   If $p(x)$ is a polynomial in $P_2(x)$, then $T(p(x))$ is the antiderivative $q(x)$ of $p(x)$ such that $q(0) = 0$. Another way to say this is

$$T(p(x)) = \int_0^x p(t)\,dt.$$

   If $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$ are the standard bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$, find the matrix $[T]_\beta^\gamma$.

   The columns of $[T]_\beta^\gamma$ are the γ-coordinates of the images of the basis vectors in β.

   The basis vectors in β are $\{1, x, x^2\}$.

   Their images are $T(1) = x$, $T(x) = \frac{1}{2}x^2$, $T(x_2) = \frac{1}{3}x^3$.

   To find the columns of the matrix, we write these vectors out as linear combinations of the vectors in γ, which gives us their γ-coordinates:

   $T(1) = x = (0)1 + (1)x + (0)x^2 + (0)x^3$,
   $T(x) = \frac{1}{2}x^2 = (0)1 + (0)x + \frac{1}{2}x^2 + (0)x^3$,
   $T(x^2) = \frac{1}{3}x^3 = (0)1 + (0)x + (0)x^2 + \frac{1}{3}x^3$.

$$[T(1)]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, [T(x)]_\gamma = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, [T(x^2)]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix}.$$

   These coordinate vectors are the columns of the matrix.

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

3. Let $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,1),(1,-1)\}$ be two ordered bases for $\mathbb{R}^2$.

(a) Find:

$$[(1,1)]_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[(1,-1)]_\beta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) Find:

$$[(1,0)]_\gamma = \begin{pmatrix} \frac{1}{2} \\ \\ \frac{1}{2} \end{pmatrix}$$

$$[(0,1)]_\gamma = \begin{pmatrix} \frac{1}{2} \\ \\ -\frac{1}{2} \end{pmatrix}$$

(c) Recall that I denotes the identity function, $I(\vec{v}) = \vec{v}$. Find the matrices:

$$[I]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[I]_\gamma^\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[I]_\beta^\gamma = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$[I]_\gamma^\beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations. Let $\beta = \{(1,0),\, (0,1)\}$ be the standard ordered basis for $\mathbb{R}^2$.

   (a) If $T(1,0) = (a,c)$ and $T(0,1) = (b,d)$, then find:

   $$T(x,y) = (ax + by,\; cx + dy)$$

   $$[T]_\beta^\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   (b) If $U(x,y) = (\overline{a}x + \overline{b}y,\; \overline{c}x + \overline{d}y)$, then find :

   $$U(1,0) = (\overline{a},\; \overline{c})$$

   $$U(0,1) = (\overline{b},\; \overline{d})$$

   $$[U]_\beta^\beta = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$$

   (c) The composition of U and T is denoted $T \circ U$, or simply TU, and is defined by $TU(x,y) = T(U(x,y))$. For T as in part (a) and U as in part (b), find:

   $$TU(1,0) = T(\overline{a},\; \overline{c}) = (a\overline{a} + b\overline{c},\; c\overline{a} + d\overline{c})$$

   $$TU(0,1) = T(\overline{b},\; \overline{d}) = (a\overline{b} + b\overline{d},\; c\overline{b} + d\overline{d})$$

   $$[TU]_\beta^\beta = \begin{pmatrix} a\overline{a} + b\overline{c} & a\overline{b} + b\overline{d} \\ c\overline{a} + d\overline{c} & c\overline{b} + d\overline{d} \end{pmatrix}$$

In the next section of the text, you will see matrix multiplication defined. *This is where the definition comes from.* Matrix multiplication is defined so that if A is the matrix of T and B is the matrix of U, then AB is the matrix of TU. You have just come up with the formula for the product of two 2×2 matrices.

5. This an extra problem, not really part of our Math 24 study.

   If we think of function composition as a kind of multiplication, then if V is a vector space over a field F, the collection $\mathcal{L}(V)$ of linear transformations from V to itself has an addition operation and a multiplication operation. (We know that $\mathcal{L}(V)$ is closed under these operations; the sum of linear functions is linear and the composition of linear functions is linear.)

   With these two operations, is $\mathcal{L}(V)$ a field? If not, which of the field axioms hold, and which do not?

   I'll leave this one as a challenge.

   This algebraic structure turns out to obey many of the field axioms, but not all of them. It is not a field but it is a *ring*. The ring axioms include many but not all of the field axioms.

   The field axioms:

   (F1) (a) Adddition is commutative.

   (b) Multiplication is commutative.

   (F2) (a) Adddition is associative.

   (b) Multiplication is associative.

   (F3) (a) There is an additive identity element.

   (b) There is a multiplicative identity element (distinct from the additive identity).

   (F4) (a) Every element has an additive inverse.

   (b) Every element except the additive identity has an multiplicative inverse.

   (F5) Multiplication distributes over addition.