Math 24

Winter 2014

Proof by Mathematical Induction: Sample Solutions

Proposition: Let V be a vector space, and $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be a linearly dependent subset of V. Then either $\vec{x}_1 = \vec{0}$, or there is some $k < n$ such that $\vec{x}_{k+1}$ is a linear combination of vectors $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$.

Reworded Proposition: Let V be a vector space. For every natural number n, for every linearly dependent subset $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ of V, either $\vec{x}_1 = \vec{0}$, or there is some $k < n$ such that $\vec{x}_{k+1}$ is a linear combination of vectors $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$.

Proof: We will prove this by induction on n.

Base Case: We must prove that for every linearly dependent subset $\{\vec{x}_1\}$ of V, either $\vec{x}_1 = \vec{0}$, or there is some $k < 1$ such that $\vec{x}_{k+1}$ is a linear combination of vectors $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$.

Inductive Step: Assume that n is a natural number such that for every linearly dependent subset $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ of V, either $\vec{x}_1 = \vec{0}$, or there is some $k < n$ such that $\vec{x}_{k+1}$ is a linear combination of vectors $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$.. (This is the inductive hypothesis.)

We must show for every linearly dependent subset $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{n+1}\}$ of V, either $\vec{x}_1 = \vec{0}$, or there is some $k < n+1$ such that $\vec{x}_{k+1}$ is a linear combination of vectors $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$.

Proposition: Let V be a vector space, and S be a finite subset of V. Then there is a linearly independent subset $L \subseteq S$ having the same span.

Reworded Proposition: Let V be a vector space. For every natural number n, for every n-element subset S of V, there is a linearly independent subset $L \subseteq S$ having the same span.

Proof: We will prove this by induction on n.

Base Case: We must prove that for every 1-element subset S of V, there is a linearly independent subset $L \subseteq S$ having the same span.

Inductive Step: Assume that n is a natural number such that for every n-element subset S of V, there is a linearly independent subset $L \subseteq S$ having the same span. (This is the inductive hypothesis.)

We must prove that for every $(n+1)$-element subset S of V, there is a linearly independent subset $L \subseteq S$ having the same span.

Proposition: Let V be a vector space that is generated by a set G containing exactly n vectors. Suppose L is a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Reworded Proposition: Let V be a vector space that is generated by a set G containing exactly n vectors. For every natural number m, for every linearly independent subset L

of V containing exactly m vectors, we have that $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.

Proof: We will prove this by induction on m.

Base Case: We must prove for every linearly independent subset L of V containing exactly 1 vector, we have that $1 \leq n$ and there exists a subset H of G containing exactly $n - 1$ vectors such that $L \cup H$ generates V.

Inductive Step: Assume that m is a natural number such that for every linearly independent subset L of V containing exactly m vectors, we have that $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.. (This is the inductive hypothesis.)

We must prove for every linearly independent subset L of V containing exactly $m + 1$ vectors, we have that $m + 1 \leq n$ and there exists a subset H of G containing exactly $n - (m + 1)$ vectors such that $L \cup H$ generates V.

Notes:

1. I used $n = 1$ as my base case because this is practice in setting up proofs in exactly this way.

   For the third proposition in particular, it might be better (and certainly makes the base case easier) to use $m = 0$ as the base case. In this case, rather than "for every natural number m," we would say "for every integer $m \geq 0$."

2. Once you start trying to prove the second and third proposition, it might be useful to prove the following lemma.

   Let S and S' be subsets of a vector space V.
   If $S' \subseteq span(S)$, then $span(S') \subseteq span(S)$.

3. From that lemma you can prove the following, perhaps more directly useful, lemma.

   Let S be a subset of a vector space V, and x and y be vectors.
   Suppose $x \in span(S \cup \{y\})$ and $y \in span(S \cup \{x\})$.
   Then $span(S \cup \{x\}) = span(S \cup \{y\})$.