Math 24
Winter
Special Assignment due Monday, January 20

Please hand in this assignment on a different piece of paper than the regular homework. Do not staple it together with the regular homework. (Be sure your name is on all assignments.)

Let V be any vector space and W be a subspace of V. For any vector x in V, we define the *coset* of W containing x to be

$$x + W = \{x + w \mid w \in W\}.$$

If V is $\mathbb{R}^2$ or $\mathbb{R}^3$, you can picture $x + W$ geometrically as a translation of W by the vector x. For example, if $V = \mathbb{R}^2$ and W is the x-axis,

$$W = \{(x, 0) \mid x \in \mathbb{R}\},$$

then the coset of W containing $(1, 1)$ is given by

$$(1,1) + W = \{(1,1) + (x,y) \mid (x,y) \in W\} = \{(1,1) + (x,0) \mid x \in \mathbb{R}\} = \{(x+1,1) \mid x \in \mathbb{R}\}.$$

That is, $(1, 1) + W$ is the line $y = 1$. In general, if W is the x-axis, then $(a, b) + W$, the coset of W containing (a, b), is the line through (a, b) parallel to the x-axis.

Assignment: Prove the following. For any vector space V and any subspace W of V:

1. For any x in V,
$$x \in (x + W).$$

2. For any x and y in V,
$$(x - y) \in W \implies (x + W) = (y + W).$$

3. For any x and y in V,
$$(x - y) \notin W \implies (x + W) \cap (y + W) = \emptyset.$$

Note: What you have shown is that the cosets of W form a *partition* of V. This means that different (unequal) cosets are disjoint (do not overlap), and together, the cosets cover all of V (every $x \in V$ belongs to some coset). In the example, the lines parallel to the x-axis form a partition of $\mathbb{R}^2$.

Note: If X and Y are sets, one way to prove $X = Y$ is to prove every element of X is in Y, and every element of Y is in X. One way to prove $X = \emptyset$ is to assume there is a element in X, and get a contradiction.