Math 24
Winter 2014
Special Assignment due Monday, February 3

Let V be any vector space and W be a subspace of V. For any vector x in V, we define the *coset* of W containing x to be

$$x + W = \{x + w \mid w \in W\}.$$

We denote the collection of cosets of W in V by V/W:

$$V/W = \{x + W \mid x \in V\}.$$

For your last assignment, you proved that addition of cosets is well-defined, where

$$(x + W) + (y + W) = (x + y) + W.$$

Assignment: Prove that V/W, with addition defined as above, satisfies vector space axioms (VS2), (VS3), and (VS4).

Note that for (VS3), for example, you should choose a specific element of V/W, and show that element is an additive identity.

As an example, here is a proof that V/W satisfies axiom (VS1).

Proposition: Let W be a subspace of a vector space V. Addition of cosets in V/W is commutative.

Proof: Let $X, Y \in V/W$. Then $X = x + W$ and $Y = y + W$ for some $x, y \in V$. By the definition of addition of cosets, we have

$$\begin{aligned} X + Y &= (x + W) + (y + W) \\ &= (x + y) + W \\ &= (y + x) + W \\ &= (y + W) + (x + W) \\ &= Y + X. \end{aligned}$$

This is what we needed to prove.

Note: In going from $(x + y) + W$ to $(y + x) + W$, I used the fact that addition of vectors is commutative without comment. At this point in the course, you can assume your reader is familiar with the vector space axioms. However, in the context of a particular proof, it may make things easier to follow if you point out where you are using the vector space axioms. Use your judgment.

Note: (This note is just cultural enrichment. You can ignore it, or read it later.) We can make a similar definition for other sorts of structures and substructures. For example, the integers $\mathbb{Z}$ with addition and multiplication form a "commutative ring with unity." This is a structure that satisfies all the axioms for a field except possibly the existence of multiplicative inverses. The set of multiplies of n

$$n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$$

is a kind of substructure of $\mathbb{Z}$ called an "ideal." This means it is closed under addition, and also under multiplication by any element of $\mathbb{Z}$. Now if we define cosets of $n\mathbb{Z}$ the same way we did above,

$$x + n\mathbb{Z} = \{x + m \mid m \in n\mathbb{Z}\},$$

we can define addition and multiplication of cosets

$$(x + n\mathbb{Z}) + (y + n\mathbb{Z}) = (x + y) + n\mathbb{Z} \quad \text{and} \quad (x + n\mathbb{Z})(y + n\mathbb{Z}) = (xy) + n\mathbb{Z}.$$

We get the structure $\mathbb{Z}/n\mathbb{Z}$, whose elements are cosets $0 + n\mathbb{Z}$, $1 + n\mathbb{Z}$, $\ldots (n-1) + n\mathbb{Z}$.

$\mathbb{Z}/2\mathbb{Z}$ is the same as $\mathbb{Z}_2$ (defined in Appendix C of the textbook), except that instead of calling the elements $0 + 2\mathbb{Z}$ and $1 + 2\mathbb{Z}$, the textbook just calls them 0 and 1. Another name for $\mathbb{Z}/n\mathbb{Z}$ is "the integers modulo n." If you're up for a challenge, you might notice that while $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are fields, $\mathbb{Z}/4\mathbb{Z}$ is not. For which n is $\mathbb{Z}/n\mathbb{Z}$ a field?