Math 24, Winter 2020, Pset 3

This problem set is due at the start of lecture on Wednesday January 29.

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and $T(1,0) = (0,1)$, $T(1,1) = (2,3)$.
   (a) Express $(1,2)$ as a linear combination of $(1,0)$ and $(1,1)$.
   (b) What is $T(1,2)$?

2. Let V, W be finite dimensional vector spaces over the same field F, and $T : V \to W$ a linear transformation. Prove that if the dimension of W is greater than the dimension of V, then T cannot be onto.

3. Let $T : V \to W$ be a linear transformation that is one-to-one. Prove that if $\{u_1, u_2, \ldots, u_n\}$ is a linearly independent set in V, then $\{T(u_1), T(u_2), \ldots, T(u_n)\}$ is linearly independent in W.

4. Let $A = [T_\theta]_\beta$ be the matrix of counterclockwise rotation $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by angle θ, using the standard basis $\beta = \{e_1, e_2\}$.
   (a) Calculate A^3.
   (b) Which formulas can you derive for $\cos(3\theta)$ and $\sin(3\theta)$ from the components of the matrix A^3? Explain your answer.
   (c) If $\theta = 2\pi/3$, what is the matrix $B = [T_\theta]_\gamma$ for the basis $\{(\sqrt{3}, 1), (-\sqrt{3}, 1)\}$ of $\mathbb{R}^2$?
   (d) Calculate the matrix B^3 using matrix multiplication, and interpret the result geometrically.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation about the axis spanned by $(1,1,1)$ by an angle of $2\pi/3$. The direction of the rotation is determined by the right-hand rule. Find the matrix $[T]_\beta$ for the standard basis $\beta = \{e_1, e_2, e_3\}$.

6. Let $P_3(\mathbb{R})$ be the vector space over $\mathbb{R}$ of polynomials of degree at most 3. Consider the linear transformation
   $$D : P_3(\mathbb{R}) \to P_3(\mathbb{R}) \qquad D(f) = f'$$
   Let $\gamma = \{f_0, f_1, f_2, f_3\}$ be the basis for $P_3(\mathbb{R})$ with $f_k(x) = \frac{1}{k!}x^k$.
   (a) Find the matrix $C = [D]_\gamma$.
   (b) Calculate C^4. What does the result say about derivatives?