Math 24, Winter 2020, Pset 4

This problem set is due on Friday February 28.

1. Let V be a finite dimensional vector space over a field F, and let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis of V. Denote by $\phi : V \to F^n$ the isomorphism

$$\phi(x) := [x]_\beta$$

   (a) If $K \subset V$ is a subspace of V, let $\phi(K) \subset F^n$ be the set

$$\phi(K) = \{y \in F^n \mid \text{there exists } x \in K \text{ such that } y = \phi(x)\}$$

   Prove that $\phi(K)$ is a subspace of F^n.

   (b) Prove that K is isomorphic to $\phi(K)$.

   (c) Prove that $\dim K = \dim \phi(K)$.

   (d) Let $T : V \to V$ be a linear map, with $n \times n$ matrix $A = [T]_\beta$. Prove that $\operatorname{rank} T = \operatorname{rank} A$.

2. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a, b) = (2a + b, a - 3b)$$

   What is the matrix $[T]_\beta$ if $\beta = \{e_1, e_2\}$ is the standard ordered basis of $\mathbb{R}^2$.

   (b) Let γ be the ordered basis

$$\gamma = \{(1, 1), (1, 2)\}$$

   What is the change of coordinate matrix Q that changes γ coordinates into β coordinates?

   (c) Find the inverse matrix Q^{-1}.

   (d) What is the change of coordinate matrix that changes standard β coordinates into γ coordinates?

   (e) Find $[T]_\gamma$. (Use Theorem 2.23)

3. Do exercise 2(b) and 2(f) on page 165 (section 3.2).

4. Do exercise 5(b) and 5(d) on page 165 (section 3.2).

5. Do exercise 2(b) and 2(d) on page 194 (section 3.4).

6. Do exercise 14, 16, 18 on page 222 (section 4.2).