Selected Solutions to Homework for Math 24 Winter 2021

As of January 11, 2021

Homework Assignment #1 (from January 8th):

1. Show that in an ordered field $\mathbf{F}$, we always have $1 > 0$. (Start by showing that $(-1)^2 = 1$.) Conclude that if $x < 0$ and $y < 0$, then $xy > 0$.

ANS: From the Proposition proved in lecture we know that additive inverses are unique and that $-a = (-1)\cdot a$ for any $a \in \mathbf{F}$. Since $-1+1=0$, it follows that $-(-1) = 1$. But $-(-1) = (-1)(-1)$. Hence $(-1)^2 = 1$. Assume to the contrary of what we want to prove, that $1 < 0$. Then $-1 > 0$. Thus $(-1)(-1) = 1 > 0$. But this contradicts our assumption. Hence $1 > 0$.

Now suppose that $x < 0$ and $y < 0$. Then $-x > 0$ and $-y > 0$. This means $(-x)(-y) > 0$. But $(-x)(-y) = (-1)x(-1)y = (-1)^2xy = 1\cdot xy = xy$. Hence $xy > 0$.

2. Is it possible to make the complex numbers $\mathbf{C}$ into an ordered field?

ANS: NO. Suppose that $\mathbf{C}$ had an ordering. Then we must have either $i > 0$ or $-i > 0$. In the first case, we would have $-1 = (i)^2 > 0$, but we saw above that $1 > 0$. So we have a contradiction. Since $(-i)^2$ is also -1, we have the same problem. So there is not way to make $\mathbf{C}$ into an ordered field.

3. If V is a vector space over $\mathbf{F}$ and $ax = \mathbf{0}$, for some $a \in \mathbf{F}$ and $x \in V$, then either $a = 0$ or $x = \mathbf{0}$. (Here $0 \in \mathbf{F}$ and $\mathbf{0}$ is the zero vector in V.)

ANS: Suppose that $a \neq 0$. Then $a^{-1}\cdot(ax) = a^{-1}\mathbf{0} = \mathbf{0}$ by Theorem 1.2(a) in the text. But $a^{-1}(ax) = (a^{-1}a)x = 1\cdot x = x$. Hence $x = \mathbf{0}$. The result follows.

4. Problem #16 in §1.2.

ANS: Since $\mathbf{Q} \subset \mathbf{R}$, $r \cdot M \in V$ for all $r \in \mathbf{Q}$. Axioms VS1–VS4 hold since they do not involve scalar multiplication. Axiom VS5 holds since $1 \in \mathbf{Q}$. Axioms VS6–VS8 hold for all $a, b \in \mathbf{Q}$ since $a, b \in \mathbf{R}$. Therefore V is a vector space over the rationals.

5. Problem #18 in §1.2.

ANS: To see that V is not a vector space, notice that $(1,0)+(2,0) = (5,0)$ and $(2,0)+(1,0) = (4,0)$. Hence Axiom VS1 is violated.

Comment: Note that you must give a specific example to show this.

6. Problem #21 in §1.2.

ANS: We have to verify VS1–VS8. For example, to verify VS1, we notice that since both V and W satisfy VS1,

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2, w_1) = (v_2, w_2) + (v_1, w_1).$$

The proof of VS2 is similar. To verify VS3, we let $0 = (0_V, 0_W)$. For VS4, let $-(v, w) = (-v, -w)$. VS5 is straightforward. For VS6

$$ab(v, w) = (abv, abw) = a(bv, bw) = a \cdot b \cdot (v, w).$$

Etc.