Math 24: Winter 2021
Lecture 1

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The Ground Rules

1. Ok, so class ends at 11:25! I am really sorry about running over—not by little, but by a lot—on Friday. All I can say is that it was a heck of a week.
2. But we need to keep going, so let's first be sure that we are recording.
3. Remember that my office hours (on zoom) are T 2-3 and Th 10-11. For details, see `math.dartmouth.edu:~dana/office_hours/`
4. But first, are there any questions from last time?

Definition

A vector space over a field $\mathbf{F}$ is a set V together with operations $(x, y) \mapsto x + y$ from $V \times V$ to V (called addition) and $(a, v) \mapsto a \cdot v$ from $\mathbf{F} \times V \to V$ (called scalar multiplication) such that the following axioms hold for all $x, y, z \in V$ and $a, b \in \mathbf{F}$.

VS1: $x + y = y + x$.

VS2: $(x + y) + z = x + (y + z)$.

VS3: There is an element $0 \in V$ such that $x + 0 = x$ for all x.

VS4: For each $x \in V$ there is a $-x \in V$ such that $-x + x = 0$.

VS5: For all $x \in V$, $1 \cdot x = x$.

VS6: $(ab) \cdot x = a \cdot (b \cdot x)$.

VS7: $a \cdot (x + y) = a \cdot x + a \cdot y$.

VS8: $(a + b) \cdot x = a \cdot x + b \cdot x$.

return1 return2 return3

The Example

Example

Let $\mathbf{F}$ be a field and $n \in \mathbf{N}$. Then the set of *n-tuples* $\mathbf{F}^n = \{\,(a_1, a_2, \dots, a_n) : a_k \in \mathbf{F}\,\}$ is a vector space with the operations $(a_1, \dots, a_n) + (a_1', \dots, a_n') = (a_1 + a_1', \dots, a_n + a_n')$ and $a \cdot (a_1, \dots, a_n) = (aa_1, \dots, aa_n)$.

Example

Let $\mathscr{F}(X, \mathbf{F})$ be the set of all functions from a set X to the field $\mathbf{F}$. Then the $\mathscr{F}(X, \mathbf{F})$ is a vector space over $\mathbf{F}$ with respect to the operations $(f+g)(x) = f(x) + g(x)$ and $(af)(x) = af(x)$ for all $x \in X$ and $a \in \mathbf{F}$. The zero element is the zero function: $z(x) = 0$ for all $x \in X$.

1. To prove that $\mathscr{F}(X, \mathbf{F})$ is a vector space, we have to carefully check that axioms VS1–VS8 all hold.
2. For example, VS1 says that addition is commutative: $f+g = g+f$ for all $f, g \in \mathscr{F}(X, \mathbf{F})$. But for any $x \in X$, $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$. This means exactly that $f+g = g+f$. Checking that addition is associative (aka VS2) is similar.

More Checking

3. To check axiom VS3, we need to see that the zero function, z, acts as a additive identity—that is, the as the zero vector 0_V in $V = \mathscr{F}(X, \mathbf{F})$ so that $f + z = f$ for any $f \in \mathscr{F}(X, \mathbf{F})$. But $(f + z)(x) = f(x) + z(x) = f(x) + 0 = f(x)$.
4. To check VS4, we need to see each $f \in \mathscr{F}(X, \mathbf{F})$ has an additive inverse $-f$ so that $-f + f = z$. But we just want $(-f)(x) = -f(x)$.
5. VS5 is almost automatic: $(1 \cdot f)(x) = 1 \cdot f(x) = f(x)$.
6. I'll leave it to you to check VS6, VS7, and VS8.
7. Thus $\mathscr{F}(X, \mathbf{F})$ is a vector space over $\mathbf{F}$ for any set X and any field $\mathbf{F}$.

Example

An element $\sigma \in \mathscr{F}(\mathbf{N}, \mathbf{F})$ is called a sequence in $\mathbf{F}$. If $a_n = \sigma(n)$, then we usually write (a_n) in place of σ. Thus the set V of sequences in $\mathbf{F}$ is a vector space over $\mathbf{F}$ with $(a_n) + (b_n) = (a_n + b_n)$ and $a \cdot (a_n) = (aa_n)$. Note that the zero element here is just the zero sequence (a_n) with $a_n = 0$ for all $n \geq 0$

Question

What if we let $V = \{ (x, y) : x, y \in \mathbf{R} \}$ but we define $(x, y) + (x', y') = (xx', yy')$ and $a \cdot (x, y) = (ax, ay)$. Is V a vector space over $\mathbf{R}$? Go

Question

Let $V = \{0\}$ be the set containing a single vector 0. Define $0 + 0 = 0$ and $a \cdot 0 = 0$ for all $a \in \mathbf{F}$. Is V a vector space over $\mathbf{F}$?

Definition

We call $V = \{0\}$ viewed as a vector space over $\mathbf{F}$ the zero vector space over $\mathbf{F}$.

Example

Let $V = \mathbf{R}^n$. Then V is our favorite example of a vector space over $\mathbf{R}$. What if we retain the usual notion of vector addition in $\mathbf{R}^n$ but let $\mathbf{F} = \mathbf{Q}$ act in the usual way by scalar multiplication. Is $\mathbf{R}^n$ then a vector space over $\mathbf{Q}$? How could we generalize this?

Break Time

- Let's Take a Break.

Some Theory

Theorem

Let V be a vector space over $\mathbf{F}$. If $x, y \in V$, then there is a unique vector $z \in V$ such that $z + x = y$.

Proof.

Fix $x, y \in V$. Let $z = y + (-x)$. (We usually write $y - x$ in place of $y + (-x)$.) Then
$z + x = (y + -x) + x = y + (-x + x) = y + 0 = y$ as required.
This proves existence.

On the other hand, if $z \in V$ is such that $z + x = y$, then
$(z + x) + (-x) = y + (-x)$. Then the left-hand side is
$z + (x + (-x)) = z + 0 = z$ and every solution must be $y - x$.
This proves uniqueness. $\square$

Corollary

If $x \in V$, then the additive inverse, $-x$ from VS4, is unique. That is if $z + x = 0$, then $z = -x$.

Proof.

This is immediate from the theorem—just let $y = 0$. $\square$

More

Theorem (Theorem 2.1 in the Text)

Let V be a vector space over $\mathbf{F}$.

1. *For all $x \in V$, $0 \cdot x = 0_V$. (Here, I have written 0_V to make it clear that 0_V is the zero vector in V and not the scalar $0 \in \mathbf{F}$.)*
2. *$a \cdot 0_V = 0_V$ for all $a \in \mathbf{F}$.*
3. *$(-a) \cdot x = -(a \cdot x) = a \cdot (-x)$ for all $a \in \mathbf{F}$ and $x \in V$. (What does each minus sign mean?)*

Proof.

(1) We have $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$. But we also have $0 \cdot x = 0_V + 0 \cdot x$. Hence $0 \cdot x = 0_V$ by the previous theorem. (Or you could just add $-0 \cdot x$ to both sides.
(2) The proof is similar to (1): $a \cdot 0_V = a \cdot (0_V + 0_V) = a \cdot 0_V + a \cdot 0_V$.
(3) Note that $(-a) \cdot x + a \cdot x = (-a + a) \cdot x = 0 \cdot x = 0_V$. Since the additive inverse is unique, we must have $(-a) \cdot x = -(a \cdot x)$ by part 1. Similarly, $a \cdot (-x) + a \cdot x = a \cdot (-x + x) = a \cdot 0_V = 0_V$ by part 2. Hence $a \cdot (-x) = -(a \cdot x)$. □

Corollary

If V is a vector space over $\mathbf{F}$, then for all $x \in V$, we have $-x = (-1) \cdot x$.

Break Time

- Time for a break and questions.

Subspaces

Definition

Let W be a subset of a vector space V over $\mathbf{F}$. We call W as subspace of V if W is a vector space over $\mathbf{F}$ with the operations of addition and scalar multiplication inherited from V.

Example

If V is a vector space over $\mathbf{F}$, then the zero subspace $\{0_V\}$ and V itself are always subspaces.

Theorem

Let V be a vector space over $\mathbf{F}$. A subset W of V is a subspace of V if and only if

1. $0_V \in W$,
2. $x, y \in W$ *implies* $x + y \in W$, *and*
3. $x \in W$ *implies* $c \cdot x \in W$ *for all* $c \in \mathbf{F}$.

Proof

Proof.

Suppose that W is a subspace of V. Then items 2 and 3 are immediate. If 0_W is the zero vector of W, then $0_W + 0_W = 0_W$. But we also have $0_V + 0_W = 0_W$. Since both equations hold in V, we must have $0_V = 0_W$ by our uniqueness result. Hence $0_V \in W$. Now suppose that items 1, 2, and 3 hold. Then VS1, VS2, VS5, VS6, VS7, and VS8 all hold because they hold in V. Go Since $0_V \in W$, we can let $0_W = 0_V$ and then VS3 holds. So we just have to check each $x \in W$ has an additive inverse in W. But $-x = (-1) \cdot x \in W$, so VS5 holds as well. □

A Great Boon to Math 24-Kind

Example

Let $W = C(\mathbf{R})$ the collection of continuous functions from $\mathbf{R}$ to $\mathbf{R}$. Then $W \subset \mathscr{F}(\mathbf{R},\mathbf{R})$. We already know, painfully, that $\mathscr{F}(\mathbf{R},\mathbf{R})$ is a vector space over $\mathbf{R}$. Since the zero function is continuous, and since the sum of continuous functions is continous, and since a multiple of a continuous function is continuous, all three conditions of our subspace theorem are met and $C(\mathbf{R})$ is a subspace of $\mathscr{F}(\mathbf{R},\mathbf{R})$. Just as importantly, this means $C(\mathbf{R})$ is a real-vector space in its own right.

Another Important Example

Example

Let $n \in \mathbf{N}$ and let $\mathrm{P}_n(\mathbf{F})$ be the set of polynomials with coefficients in $\mathbf{F}$ that have degree at most n. Clearly, $\mathrm{P}_n(\mathbf{F}) \subset \mathrm{P}(\mathbf{F})$. Since the zero polynomial has degree -1, it belongs to $\mathrm{P}_n(\mathbf{F})$. Furthermore, the sum of two polynomials of degree at most n is a polynomial of degree at most n. Similarly, a multiple of a polynomial of degree at most n has degree at most n. Hence $\mathrm{P}_n(\mathbf{F})$ is a subspace of $\mathrm{P}(\mathbf{F})$ and hence its a vector space over $\mathbf{F}$.

Question

What if we looked as the subset W of polynomials of degree equal to n (and we're careful and include the zero polynomial as well)? Is W a subspace?

Enough

1. That is enough for today.