# Math 24: Winter 2021
# Lecture 3

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# Let's Get Started

1. We should be recording.
2. Those of you watching the videos should give me some feedback via email as to whether you prefer the zoom link to the panopto link.
3. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
4. The first homework assignment was due today via gradescope. Since this was the first assignment, you can turn it in until 10pm this evening.
5. But first, are there any questions from last time?

# Some Review

**Definition**

Let $W$ be a subset of a vector space $V$ over $\mathbf{F}$. We call $W$ as subspace of $V$ if $W$ is a vector space over $\mathbf{F}$ with the operations of addition and scalar multiplication inherited from $V$.

**Example (Low Hanging Friut)**

If $V$ is a vector space over $\mathbf{F}$, then the zero subspace $\{0_V\}$ and $V$ itself are always subspaces.

**Theorem**

*Let $V$ be a vector space over $\mathbf{F}$. A subset $W$ of $V$ is a subspace of $V$ if and only if*

1. $0_V \in W$,
2. $x, y \in W$ *implies* $x + y \in W$, *and*
3. $x \in W$ *implies* $c \cdot x \in W$ *for all* $c \in \mathbf{F}$.

# Proof

**Proof.**

Suppose that $W$ is a subspace of $V$. Then items 2 and 3 are immediate. If $0_W$ is the zero vector of $W$, then $0_W + 0_W = 0_W$. But we also have $0_V + 0_W = 0_W$. Since both equations hold in $V$, we must have $0_V = 0_W$ by our uniqueness result. Hence $0_V \in W$. Now suppose that items 1, 2, and 3 hold. Then VS1, VS2, VS5, VS6, VS7, and VS8 all hold because they hold in $V$. Go Since $0_V \in W$, we can let $0_W = 0_V$ and then VS3 holds. So we just have to check each $x \in W$ has an additive inverse in $W$. But $-x = (-1) \cdot x \in W$, so VS5 holds as well. $\square$

# A Great Boon to Math 24-Kind

## Example (Last Time)

Let $W = C(\mathbf{R})$ the collection of continuous functions from $\mathbf{R}$ to $\mathbf{R}$. Then $W \subset \mathscr{F}(\mathbf{R},\mathbf{R})$. We already know, painfully, that $\mathscr{F}(\mathbf{R},\mathbf{R})$ is a vector space over $\mathbf{R}$. Since the zero function is continuous, and since the sum of continuous functions is continous, and since a multiple of a continuous function is continuous, all three conditions of our subspace theorem are met and $C(\mathbf{R})$ is a subspace of $\mathscr{F}(\mathbf{R},\mathbf{R})$. Just as importantly, this means $C(\mathbf{R})$ is a real-vector space in its own right.

# Polynomials

## Remark

To make our life easier—because Math 24 is complicated enough—unless specifically told otherwise we will assume that our field $\mathbf{F}$ is such that polynomials $p(x) = a_0 + a_1 x + \cdots a_n x^n \in \mathrm{P}(\mathbf{F})$ can be identified with the corresponding function from $\mathbf{F} \to \mathbf{F}$. This is true for all subfields of $\mathbf{R}$ or for $\mathbf{C}$. Therefore $\mathrm{P}(\mathbf{F}) \subset \mathscr{F}(\mathbf{F}, \mathbf{F})$. It is not hard to see that $\mathrm{P}(\mathbf{F})$ is a subspace! Clearly the zero polynomial is the just the zero function. Similarly, addition of polynomials corresponds to addition of functions and scalar multiplication corresponds to scalar multiplication of functions.

# Another Important Example

**Example**

Let $n \in \mathbf{N}$ and let $\mathrm{P}_n(\mathbf{F})$ be the set of polynomials with coefficients in $\mathbf{F}$ that have degree at most $n$. Clearly, $\mathrm{P}_n(\mathbf{F}) \subset \mathrm{P}(\mathbf{F})$. Since the zero polynomial has degree $-1$, it belongs to $\mathrm{P}_n(\mathbf{F})$. Furthermore, the sum of two polynomials of degree at most $n$ is a polynomial of degree at most $n$. Similarly, a multiple of a polynomial of degree at most $n$ has degree at most $n$. Hence $\mathrm{P}_n(\mathbf{F})$ is a subspace of $\mathrm{P}(\mathbf{F})$ and hence its a vector space over $\mathbf{F}$.

**Question**

What if we looked as the subset $W$ of polynomials of degree equal to $n$ (and we're careful and include the zero polynomial as well)? Is $W$ a subspace?

# The Transpose

**Definition**

If $A = (A_{ij})$ is a $m \times n$ matrix, then the transpose of $A$ is the $n \times m$ matrix $A^t = (B_{ij})$ where $B_{ij} = A_{ji}$.

**Example**

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$. Then $A^t = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$.

**Example**

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Also $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

# Symmetric Matrices

**Definition**

A $n \times n$ matrix $A$ is called symmetric if $A = A^t$. (This only makes sense for square matrices!)

**Lemma**

*If $A$ and $B$ are $m \times n$ matrices and $a, b \in \mathbf{F}$, then* $(aA + bB)^t = aA^t + bB^t$.

**Proof.**

This is a homework exercise. $\square$

**Proposition**

*The set $W$ of symmetric matrices in $M_{n\times n}(\mathbf{F})$ is a subspace.*

**Proof.**

We just appeal to our theorem on subspaces. First, the zero matrix $O$ is symmetric, so $O \in W$. If $A, B \in W$, then $(A+B)^t = A^t + B^t = A + B$, so $A + B \in W$. If $A \in W$ and $a \in \mathbf{F}$, then $(aA)^t = aA^t = aA$, so $aA \in W$. Thus $W$ is a subspace. □

# More Types of Matrices

## Definition

Let $A = (A_{ij})$ be a $n \times n$ matrix. We say that $A$ is upper triangular if $A_{ij} = 0$ whenever $i > j$. We say that $A$ is lower triangular if $A_{ij} = 0$ whenever $i < j$. We say that $A$ is diagonal if $A_{ij} = 0$ if $i \neq j$.

## Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A$ is upper triangular, $B$ is lower triangular, while $C$ is diagonal as well as both upper and lower triangular.

**Proposition**

*The set $W$ of upper triangular matrices in $M_{n\times n}(\mathbf{F})$ is a subspace as is the set $W'$ of lower triangular matrices in $M_{n\times n}(\mathbf{F})$.*

**Proof.**

Note that the zero matrix $O$ is upper triangular. Hence $O \in W$. If $A, B \in W$ and $i > j$, then $(A+B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$, so $A + B \in W$. Similarly, $aA \in W$ if $A \in W$. Thus $W$ is a subspace. The proof for $W'$ is similar. $\square$

# Break Time

Let's take a short break to rest, catch up, and ask questions.

# Examples

## Example

Let $W = \{\, (a_1, a_2, a_3) \in \mathbf{R}^3 : 2a_1 = a_3 \,\}$. Is $W$ as subspace of $\mathbf{R}^3$?

## Solution

*Yes! Clearly the zero vector,* $(0, 0, 0) \in W$. *If* $(a_1, a_2, a_3), (a_1', a_2', a_3') \in W$, *then* $(a_1, a_2, a_3) + (a_1', a_2', a_3') = (a_1 + a_1', a_2 + a_2', a_3 + a_3')$. *Then* $2(a_1 + a_1') = 2a_1 + 2a_1' = a_3 + a_3'$. *Therefore* $(a_1, a_2, a_3) + (a_1', a_2', a_3') \in W$. *Similarly, for any* $a \in \mathbf{R}$, $a(a_1, a_2, a_3) \in W$ *if* $(a_1, a_2, a_3) \in W$.

## Example

Let $W' = \{\, (a_1, a_2, a_3) \in \mathbf{R}^3 : 2a_1 = a_3 + 1 \,\}$. Is $W'$ as subspace of $\mathbf{R}^3$?

## Solution

*No!* $(0, 0, 0) \notin W'$.

# Intersections

## Theorem

*The intersection of subspaces of a vector space $V$ is a subspace of $V$.*

## Remark

There is no limit to the number of subspaces considered here. Sometimes we even emphasize this by saying that the "arbitrary intersection of subspaces is a subspace".

## Proof.

Let $\mathscr{C}$ be a collection of subspaces of $V$. Let $W$ be the intersection—$W = \bigcap_{W' \in \mathscr{C}} W'$. Since $0_V \in W'$ for any $W' \in \mathscr{C}$, we have $0_V \in W$. Now let $x, y \in W$ and $a \in \mathbf{F}$. Then for all $W' \in \mathscr{C}$, we have $x, y \in W'$. Hence $x + y \in W'$ and $ax \in W'$. Hence $x + y$ and $ax \in W$. This shows $W$ is a subspace. $\square$

# Example

## Example

Let $W_1$ be the subspace of $M_{n\times n}(\mathbf{F})$ consisting of upper triangular matrices and $W_2$ the subspace of lower triangular matrices. Since $W_1 \cap W_2$ is the set of diagonal matrices, we see immediately, that the set of diagonal matrices is a subspace as well. Of course, this is easy to prove directly.

**Example**

Let $W_1 = \{ (x, y) \in \mathbf{R}^2 : y = 0 \}$ and let $W_2 = \{ (x, y) \in \mathbf{R}^2 : x = 0 \}$. You can easily check that $W_1$ and $W_2$ are subspaces. What subspace is $W_1 \cap W_2$? Is the union $W_1 \cup W_2$ a subspace?

**Solution**

*Clearly, $W_1 \cap W_2 = \{ 0_{\mathbf{R}^2} \}$ is the zero subspace. But $W_1 \cup W_2$ is just the union of the coordinate axes. Note that $(1, 0) \in W_1$ and $(0, 1) \in W_2$, but $(1, 1) = (1, 0) + (0, 1) \notin W_1 \cup W_2$. So the union of two subspaces need not be a subspace.*

# Unions Again

**Remark**

Since the union of two subspaces $W_1$ and $W_2$ need not be a subspace, might want to find the "smallest" subspace $W$ that contains both $W_1$ and $W_2$. But is there a smallest such subspace?

**Definition**

If $S_1$ and $S_2$ are subsets of a vector space $V$, then we define

$$S_1 + S_2 = \{\, x + y : x \in S_1 \text{ and } y \in S_2 \,\}.$$

**Proposition**

*Suppose that $W_1$ and $W_2$ are subspaces of a vector space $V$. Then $W_1 + W_2$ is a subspace of $V$ containing $W_1 \cup W_2$ and is contained in every subspace of $V$ that contains $W_1 \cup W_2$. That is, $W_1 + W_2$ is the smallest subspace containing both $W_1$ and $W_2$.*

**Proof.**

You will prove this for homework. □

# Direct Sums

**Definition**

Let $W_1$ and $W_2$ be subspaces of a vector space $V$. We say $V$ is the direct sum of $W_1$ and $W_2$ if $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. In this case we write $V = W_1 \oplus W_2$.

**Example**

Let $V = \mathbf{R}^2$, $W_1 = \{\, (x, 0) : x \in \mathbf{R} \,\}$, and $W_2 = \{\, (0, y) : y \in \mathbf{R} \,\}$. Then $\mathbf{R}^2 = W_1 \oplus W_2$.

**Remark**

In our next example, we need to know that if $a \in \mathbf{F}$ and $a = -a$, then $a = 0$. Note that $a = -a$ implies $(1 + 1) \cdot a = 0$. So we're good if $1 + 1 \neq 0$. The fancy way to ensure this—other than just insisting that we work with subfields of $\mathbf{R}$ or $\mathbf{C}$—is to say that "$\mathbf{F}$ does not have characteristic 2". Then $1 + 1 = 2 \neq 0$ and $\frac{1}{2}$ makes sense.

# Functions

## Example

Let $V = \mathscr{F}(\mathbf{F}, \mathbf{F})$ and assume $\mathbf{F}$ does not have characteristic 2. We say that $f \in V$ is even if $f(-a) = f(a)$ for all $a \in \mathbf{F}$ and we say that $f$ is odd if $f(-a) = -f(a)$ for all $a \in \mathbf{F}$. For example, $f(a) = a^2$ is even while $f(a) = a^3$ is odd. I will leave it to you to check that the set $W_e$ of even functions and the set $W_o$ of odd functions are both subspaces. Suppose that $f \in W_e \cap W_o$. Then for all $a \in \mathbf{F}$, $f(-a) = f(a) = -f(a)$. Since $\mathbf{F}$ is nice, this forces $f$ to be the zero function so $W_e \cap W_o = \{0\}$. On the other hand, given $f \in V$, we have $f = f_e + f_o$ with $f_e(a) = \frac{1}{2}(f(a) + f(-a))$ and $f_o(a) = \frac{1}{2}(f(a) - f(-a))$. But $f_e \in W_e$ and $f_o \in W_o$. Therefore $V = W_e \oplus W_o$.

# Enough

1. That is enough for today.

# Recall

## Definition

A vector space over a field $\mathbf{F}$ is a set $V$ together with operations $(x, y) \mapsto x + y$ from $V \times V$ to $V$ (called addition) and $(a, v) \mapsto a \cdot v$ from $\mathbf{F} \times V \to V$ (called scalar multiplication) such that the following axioms hold for all $x, y, z \in V$ and $a, b \in \mathbf{F}$.

VS1: $x + y = y + x$.

VS2: $(x + y) + z = x + (y + z)$.

VS3: There is an element $0 \in V$ such that $x + 0 = x$ for all $x$.

VS4: For each $x \in V$ there is a $-x \in V$ such that $-x + x = 0$.

VS5: For all $x \in V$, $1 \cdot x = x$.

VS6: $(ab) \cdot x = a \cdot (b \cdot x)$.

VS7: $a \cdot (x + y) = a \cdot x + a \cdot y$.

VS8: $(a + b) \cdot x = a \cdot x + b \cdot x$.

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