Math 24: Winter 2021
Lecture 4

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Friday January 13, 2021

Let's Get Started

1. We should be recording.
2. Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
3. There is no class on Monday (January 18th) due to the MLK holiday.
4. The second homework assignment is due Wednesday via `gradescope` by 10am.
5. But first, are there any questions from last time?

Review

Proposition (Homework)

Suppose that W_1 and W_2 are subspaces of a vector space V. Then $W_1 + W_2$ is a subspace of V containing $W_1 \cup W_2$ and is contained in every subspace of V that contains $W_1 \cup W_2$. That is, $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2.

Definition

Let W_1 and W_2 be subspaces of a vector space V. We say V is the direct sum of W_1 and W_2 if $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. In this case we write $V = W_1 \oplus W_2$.

Good Fields

Remark

Most of the time, we sill assume that our field **F** is a subfield of the real numbers or the complex numbers. But it is fun to play with a general field when we can. However, it our next example, we need to know that if $a \in \mathbf{F}$ and $a = -a$, then $a = 0$. Note that $a = -a$ implies $(1 + 1) \cdot a = 0$. So we're good if $1 + 1 \neq 0$. The fancy way to ensure this—other than just insisting that we work with subfields of **R** or **C**—is to say that "**F** does not have characteristic 2". Then $1 + 1 = 2 \neq 0$ and $\frac{1}{2}$ makes sense.

Example

Let $V = \mathscr{F}(\mathbf{F}, \mathbf{F})$ and assume $\mathbf{F}$ does not have characteristic 2. We say that $f \in V$ is even if $f(-a) = f(a)$ for all $a \in \mathbf{F}$ and we say that f is odd if $f(-a) = -f(a)$ for all $a \in \mathbf{F}$. For example, $f(a) = a^2$ is even while $f(a) = a^3$ is odd. I will leave it to you to check that the set W_e of even functions and the set W_o of odd functions are both subspaces. Suppose that $f \in W_e \cap W_o$. Then for all $a \in \mathbf{F}$, $f(-a) = f(a) = -f(a)$. Since $\mathbf{F}$ is nice, this forces f to be the zero function so $W_e \cap W_o = \{0\}$. On the other hand, given $f \in V$, we have $f = f_e + f_o$ with $f_e(a) = \frac{1}{2}(f(a) + f(-a))$ and $f_o(a) = \frac{1}{2}(f(a) - f(-a))$. But $f_e \in W_e$ and $f_o \in W_o$. Therefore $V = W_e \oplus W_o$.

Linear Combinations

Definition

Let S be a subset of a vector space V. Then we say that $v \in V$ is a linear combination of vectors from S is there are finitely many vectors, say $v_1, \dots, v_n$, in S, and scalars $a_1, \dots, a_n \in \mathbf{F}$ such that $v = a_1 v_1 + \cdots + a_n v_n$. We also say that v is linear combination of $v_1, \dots, v_n$ and call the scalars u_k the coefficients of the linear combination.

Example

Since $0v = 0_V$ for any $v \in V$, the zero vector 0_V is a linear combination of any nonempty set $S \subset V$.

Example

Since $(x, y) = x(1, 0) + y(0, 1)$, every vector in $\mathbf{F}^2$ can be obtained as a linear combination from $S = \{ (1, 0), (0, 1) \}$.

An Example

Example

Suppose that $p_1(x) = x^2 + 2x + 1$, $p_2(x) = x^2 - x + 1$, and $p_3(x) = 2x^2 + x + 2$ are in $\mathrm{P}(\mathbf{R})$. Can $f(x) = x^2 + x + 1$ be expressed as a linear combination of p_1, p_2, and p_3? return

Solution

Let's try f first. We want to find scalars $a_1, a_2, a_3 \in \mathbf{R}$ such that $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = f(x)$. Collecting terms on the left-hand side, we want

$$(a_1 + a_2 + 2a_3)x^2 + (2a_1 - a_2 + a_3)x + (a_1 + a_2 + 2a_3) = x^2 + x + 1.$$

Since two polynomials are equal exactly when their coefficients are equal, this gives the following system of linear equations:

Example

Solution

Solution (Solution Continued)

$$\begin{aligned} a_1 + a_2 + 2a_3 &= 1 \\ 2a_1 - a_2 + a_3 &= 1 \\ a_1 + a_2 + 2a_3 &= 1. \end{aligned}$$

Then, as we showed using the document camera, using "standard techniques", the solutions to the above system are the same as the solutions to

$$\begin{aligned} a_1 + \qquad a_3 &= \tfrac{2}{3} \\ a_2 + a_3 &= \tfrac{1}{3} \\ 0 &= 0. \end{aligned}$$

Now we can pick a_3 as we please and $a_1 = \frac{2}{3} - a_3$ while $a_2 = \frac{1}{3} - a_3$. Hence f is a linear combination and we get specific coefficients by taking $(a_1, a_2, a_3) \in \{\, (\frac{2}{3} - t, \frac{1}{3} - t, t) : t \in \mathbf{R} \,\}$.

Standard Techniques

If we have a system of linear equations, then we get a system with the same solutions if we

1. Interchange two equations.
2. Multiply and equation by a nonzero scalar.
3. Add a multiple of one equation to a different equation.

Then our goal is the following.

1. Arrange that the first nonzero coefficient in any equation is 1. We call this a leading coefficient.
2. If an unknown corresponds to a leading coefficient in one equation, it has zero coefficient in every other equation.
3. The subscript of a leading coefficient an equation is always larger than that in any equation above it.

Back to Our Example

Example

Let's return to the previous example, but replace $f(x) = x^2 + x + 1$ with $g(x) = x^2 + x + 2$ and ask if g is a linear combination of p_1, p_2, and p_3.

Solution

Proceeding as before, we get the system

$$\begin{aligned} a_1 + a_2 + 2a_3 &= 1 \\ 2a_1 - a_2 + a_3 &= 1 \\ a_1 + a_2 + 2a_3 &= 2. \end{aligned}$$

Solution

Solution

Then, as we showed using the document camera, using "standard techniques", the solutions to the above system are the same as the solutions to

$$\begin{aligned} a_1 + a_2 + 2a_3 &= 1 \\ -3a_2 - 3a_3 &= -1 \\ 0 &= 1. \end{aligned}$$

Since this system clearly has no solutions, neither does the original system. Hence g is not a linear combination of p_1, p_2, and p_3.

Break Time

Time for a quick break to relax and ask questions.

Span

Definition

Let S be a nonempty subset of a vector space V. Then Span(S) is defined to be the set of all linear combinations of vectors from S and is called the span of S. We also define $\text{Span}(\emptyset) = \{0_V\}$.

Example

Let $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \subset M_{2\times 2}(\mathbf{R})$. Then $a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $\text{Span}(S) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbf{R} \right\}$.

Span Theorem

Theorem

If S is any subset of a vector space V, then $\mathrm{Span}(S)$ is subspace of V containing S. Furthermore, any subspace of V containing S must also contain $\mathrm{Span}(S)$. Hence $\mathrm{Span}(S)$ is the smallest subspace of V containing S.

Proof.

If $S = \emptyset$, then $\mathrm{Span}(S) := \{0\}$. Thus $\mathrm{Span}(S)$ is a subspace V and is contained in every subspace of V.
Now suppose $S \neq \emptyset$. Then there is some $x \in S$ and hence $0 \cdot x = 0_V \in \mathrm{Span}(S)$. Now suppose that $x, y \in \mathrm{Span}(S)$. Then there are vectors $u_1, \dots, u_n$ and $v_1, \dots, v_m$ in S and scalars a_k and b_k such that $x = a_1 u_1 + \cdots + a_n u_n$ and $y = b_1 v_1 + \cdots + b_m v_m$. Since $0 \cdot v = 0_V$ for any $v \in V$, we can add 0_V to both x and y and assume that $n = m$ and that $u_k = v_k$ for $1 \leq k \leq n$. Then
$x + y = (a_1 + b_1)u_1 + \cdots + (a_n + b_n)u_n \in \mathrm{Span}(S)$. Similarly,
$a \cdot x = aa_1 u_1 + \cdots + aa_n u_n \in V$.
This shows that $\mathrm{Span}(S)$ is a subspace.

Proof

Proof Continued.

Now suppose that W is a subspace such that $S \subset W$. If $u_1, \dots, u_n \in S$ and a_k are scalars, then $a_1 u_1 + \cdots + a_n u_n \in W$ since W is closed under addition and scalar multiplication. Hence $\operatorname{Span}(S) \subset W$. □

Definition

A subset $S \subset V$ spans V or *generates* V if $V = \operatorname{Span}(S)$. In this case, we say that vectors of S span or generate V.

Example

Let W be the set of symmetric 2×2-matrices. Let $S = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}$. Then if $\begin{pmatrix} a & c \\ c & b \end{pmatrix} \in W$, then $\begin{pmatrix} a & c \\ c & b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence S spans W.

An Example

Example

Suppose that $p_1(x) = x^2 + 2x + 1$, $p_2(x) = x^2 - x + 1$, and $p_3(x) = 2x^2 + x + 2$ are in $\mathsf{P}_2(\mathbf{R})$. We showed earlier that $g(x) = x^2 + x + 2$ is not a linear combination of p_1, p_2, p_3. Hence $\{ p_1, p_2, p_3 \}$ does not span all of $P_2(\mathbf{R})$.

Break Time

Time for a short break and few questions.

An Example

Proposition

Suppose that $S_1 \subset S_2$ in a vector space V. Then $\mathrm{Span}(S_1) \subset \mathrm{Span}(S_2)$.

Proof.

If $S_1 = \emptyset$, the result is automatic (since $\{0_V\}$ is a subspace of any subspace). Otherwise, if $x \in \mathrm{Span}(S_1)$ then there are vectors $u_1, \dots, u_n \in S_1$ and scalars a_k such that $x = a_1 u_1 + \cdots + a_n u_n$. But $u_1, \dots, u_n$ are also in S_2. Hence $x \in \mathrm{Span}(S_2)$. This shows $\mathrm{Span}(S_1) \subset \mathrm{Span}(S_2)$ as required. □

Another Result

Proposition

Let $v_1, \ldots, v_{n+1}$ *be vectors in a vector space* V*. Suppose that* $v_{n+1} \in \operatorname{Span}(\{ v_1, \ldots, v_n \})$*. Then*

$$\operatorname{Span}(\{ v_1, \ldots, v_n \}) = \operatorname{Span}(\{ v_1, \ldots, v_{n+1} \}).$$

Proof.

By the previous result, $\operatorname{Span}(\{ v_1, \ldots, v_n \}) \subset \operatorname{Span}(\{ v_1, \ldots, v_{n+1} \})$. Now suppose that $x \in \operatorname{Span}(\{ v_1, \ldots, v_{n+1} \})$. Then there are scalars a_k such that $x = a_1 v_1 + \cdots a_n v_n + a_{n+1} v_{n+1}$. But assumption there are also scalars b_k such that $v_{n+1} = b_1 v_1 + \cdots + b_n v_n$. Therefore, $x = (a_1 + a_{n+1} b_1) v_1 + \cdots (a_n + a_{n+1} b_n) v_n \in \operatorname{Span}(\{ v_1, \ldots, v_n \})$. This shows that $\operatorname{Span}(\{ v_1, \ldots, v_{n+1} \}) \subset \operatorname{Span}(\{ v_1, \ldots, v_n \})$ so the sets must be equal. □

Enough

1. That is enough for today.