

Math 24: Winter 2021

Lecture 4

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Let's Get Started

- 1 We should be recording.
- 2 Remember, it is more comfortable for me if you turn on your video so that I feel like I am talking to real people.
- 3 There is no class on Monday (January 18th) due to the MLK holiday.
- 4 The second homework assignment is due Wednesday via gradescope by 10am.
- 5 But first, are there any questions from last time?

Proposition (Homework)

Suppose that W_1 and W_2 are subspaces of a vector space V . Then $W_1 + W_2$ is a subspace of V containing $W_1 \cup W_2$ and is contained in every subspace of V that contains $W_1 \cup W_2$. That is, $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2 .

Definition

Let W_1 and W_2 be subspaces of a vector space V . We say V is the **direct sum** of W_1 and W_2 if $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. In this case we write $V = W_1 \oplus W_2$.

Remark

Most of the time, we still assume that our field \mathbf{F} is a subfield of the real numbers or the complex numbers. But it is fun to play with a general field when we can. However, in our next example, we need to know that if $a \in \mathbf{F}$ and $a = -a$, then $a = 0$. Note that $a = -a$ implies $(1 + 1) \cdot a = 0$. So we're good if $1 + 1 \neq 0$. The fancy way to ensure this—other than just insisting that we work with subfields of \mathbf{R} or \mathbf{C} —is to say that “ \mathbf{F} does not have characteristic 2”. Then $1 + 1 = 2 \neq 0$ and $\frac{1}{2}$ makes sense.

Example

Let $V = \mathcal{F}(\mathbf{F}, \mathbf{F})$ and assume \mathbf{F} does not have characteristic 2. We say that $f \in V$ is **even** if $f(-a) = f(a)$ for all $a \in \mathbf{F}$ and we say that f is **odd** if $f(-a) = -f(a)$ for all $a \in \mathbf{F}$. For example, $f(a) = a^2$ is even while $f(a) = a^3$ is odd. I will leave it to you to check that the set W_e of even functions and the set W_o of odd functions are both subspaces. Suppose that $f \in W_e \cap W_o$. Then for all $a \in \mathbf{F}$, $f(-a) = f(a) = -f(a)$. Since \mathbf{F} is nice, this forces f to be the zero function so $W_e \cap W_o = \{0\}$. On the other hand, given $f \in V$, we have $f = f_e + f_o$ with $f_e(a) = \frac{1}{2}(f(a) + f(-a))$ and $f_o(a) = \frac{1}{2}(f(a) - f(-a))$. But $f_e \in W_e$ and $f_o \in W_o$. Therefore $V = W_e \oplus W_o$.

Linear Combinations

Definition

Let S be a subset of a vector space V . Then we say that $v \in V$ is a **linear combination** of vectors from S if there are finitely many vectors, say v_1, \dots, v_n , in S , and scalars $a_1, \dots, a_n \in \mathbf{F}$ such that $v = a_1 v_1 + \dots + a_n v_n$. We also say that v is linear combination of v_1, \dots, v_n and call the scalars a_k the **coefficients** of the linear combination.

Example

Since $0 \cdot v = 0_V$ for any $v \in V$, the zero vector 0_V is a linear combination of any nonempty set $S \subset V$.

Example

Since $(x, y) = x(1, 0) + y(0, 1)$, every vector in \mathbf{F}^2 can be obtained as a linear combination from $S = \{(1, 0), (0, 1)\}$.

An Example

Example

Suppose that $p_1(x) = x^2 + 2x + 1$, $p_2(x) = x^2 - x + 1$, and $p_3(x) = 2x^2 + x + 2$ are in $P(\mathbf{R})$. Can $f(x) = x^2 + x + 1$ be expressed as a linear combination of p_1 , p_2 , and p_3 ? [▶ return](#)

Solution

We want to find scalars $a_1, a_2, a_3 \in \mathbf{R}$ such that $a_1p_1(x) + a_2p_2(x) + a_3p_3(x) = f(x)$. Collecting terms on the left-hand side, we want

$$(a_1 + a_2 + 2a_3)x^2 + (2a_1 - a_2 + a_3)x + (a_1 + a_2 + 2a_3) = x^2 + x + 1.$$

Since two polynomials are equal exactly when their coefficients are equal, this gives the following system of linear equations:

Solution (Solution Continued)

$$a_1 + a_2 + 2a_3 = 1$$

$$2a_1 - a_2 + a_3 = 1$$

$$a_1 + a_2 + 2a_3 = 1.$$

Then, as we showed using the document camera, using “standard techniques”, the solutions to the above system are the same as the solutions to

$$a_1 + a_3 = \frac{2}{3}$$

$$a_2 + a_3 = \frac{1}{3}$$

$$0 = 0.$$

Now we can pick a_3 as we please and $a_1 = \frac{2}{3} - a_3$ while $a_2 = \frac{1}{3} - a_3$. Hence f is a linear combination and we get specific coefficients by taking $(a_1, a_2, a_3) \in \{(\frac{2}{3} - t, \frac{1}{3} - t, t) : t \in \mathbf{R}\}$.

Standard Techniques

If we have a system of linear equations, then we get a system with the same solutions if we

- 1 Interchange two equations.
- 2 Multiply an equation by a **nonzero** scalar.
- 3 Add a multiple of one equation to a different equation.

Then our goal is the following.

- 1 Arrange that the first **nonzero** coefficient in any equation is 1. We call this a **leading coefficient**.
- 2 If an unknown corresponds to a leading coefficient in one equation, it has zero coefficient in every other equation.
- 3 The subscript of a leading coefficient in an equation is always larger than that in any equation above it.

Example

Let's return to the [previous example](#), but replace $f(x) = x^2 + x + 1$ with $g(x) = x^2 + x + 2$ and ask if g is a linear combination of p_1 , p_2 , and p_3 .

Solution

Proceeding as before, we get the system

$$a_1 + a_2 + 2a_3 = 1$$

$$2a_1 - a_2 + a_3 = 1$$

$$a_1 + a_2 + 2a_3 = 2.$$

Solution

Then, as we showed using the document camera, using “standard techniques”, the solutions to the above system are the same as the solutions to

$$a_1 + a_2 + 2a_3 = 1$$

$$-3a_2 - 3a_3 = -1$$

$$0 = 1.$$

*Since this system clearly has no solutions, neither does the original system. Hence g is **not** a linear combination of p_1 , p_2 , and p_3 .*

Time for a quick break to relax and ask questions.

Definition

Let S be a nonempty subset of a vector space V . Then $\text{Span}(S)$ is defined to be the set of all linear combinations of vectors from S and is called the **span** of S . We also define $\text{Span}(\emptyset) = \{0_V\}$.

Example

Let $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \subset M_{2 \times 2}(\mathbf{R})$. Then
 $a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $\text{Span}(S) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbf{R} \right\}$.

Span Theorem

Theorem

If S is any subset of a vector space V , then $\text{Span}(S)$ is subspace of V containing S . Furthermore, any subspace of V containing S must also contain $\text{Span}(S)$. Hence $\text{Span}(S)$ is the smallest subspace of V containing S .

Proof.

If $S = \emptyset$, then $\text{Span}(S) := \{0\}$. Thus $\text{Span}(S)$ is a subspace V and is contained in every subspace of V .

Now suppose $S \neq \emptyset$. Then there is some $x \in S$ and hence $0 \cdot x = 0_V \in \text{Span}(S)$. Now suppose that $x, y \in \text{Span}(S)$. Then there are vectors u_1, \dots, u_n and v_1, \dots, v_m in S and scalars a_k and b_k such that $x = a_1 u_1 + \dots + a_n u_n$ and $y = b_1 v_1 + \dots + b_m v_m$. Since $0 \cdot v = 0_V$ for any $v \in V$, we can add 0_V to both x and y and assume that $n = m$ and that $u_k = v_k$ for $1 \leq k \leq n$. Then

$x + y = (a_1 + b_1)u_1 + \dots + (a_n + b_n)u_n \in \text{Span}(S)$. Similarly, $a \cdot x = aa_1 u_1 + \dots + aa_n u_n \in \text{Span}(S)$.

This shows that $\text{Span}(S)$ is a subspace. Further, if $v \in S$, then $v = 1 \cdot v \in \text{Span}(S)$ so $S \subset \text{Span}(S)$ as claimed.

Proof Continued.

Now suppose that W is a subspace such that $S \subset W$. If $u_1, \dots, u_n \in S$ and a_k are scalars, then $a_1 u_1 + \dots + a_n u_n \in W$ since W is closed under addition and scalar multiplication. Hence $\text{Span}(S) \subset W$. \square

Definition

A subset $S \subset V$ **spans** V or **generates** V if $V = \text{Span}(S)$. In this case, we say that vectors of S span or generate V .

Example

Let W be the set of symmetric 2×2 -matrices. Let $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. Then if $\begin{pmatrix} a & c \\ c & b \end{pmatrix} \in W$, then $\begin{pmatrix} a & c \\ c & b \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence S spans W .

Example

Suppose that $p_1(x) = x^2 + 2x + 1$, $p_2(x) = x^2 - x + 1$, and $p_3(x) = 2x^2 + x + 2$ are in $P_2(\mathbf{R})$. We showed earlier that $g(x) = x^2 + x + 2$ is not a linear combination of p_1, p_2, p_3 . Hence $\{p_1, p_2, p_3\}$ does not span all of $P_2(\mathbf{R})$.

Time for a short break and few questions.

An Example

Proposition

Suppose that $S_1 \subset S_2$ in a vector space V . Then $\text{Span}(S_1) \subset \text{Span}(S_2)$.

Proof.

If $S_1 = \emptyset$, the result is automatic (since $\{0_V\}$ is a subspace of any subspace). Otherwise, if $x \in \text{Span}(S_1)$ then there are vectors $u_1, \dots, u_n \in S_1$ and scalars a_k such that $x = a_1 u_1 + \dots + a_n u_n$. But u_1, \dots, u_n are also in S_2 . Hence $x \in \text{Span}(S_2)$. This shows $\text{Span}(S_1) \subset \text{Span}(S_2)$ as required. \square

Another Result

Proposition

Let v_1, \dots, v_{n+1} be vectors in a vector space V . Suppose that $v_{n+1} \in \text{Span}(\{v_1, \dots, v_n\})$. Then

$$\text{Span}(\{v_1, \dots, v_n\}) = \text{Span}(\{v_1, \dots, v_{n+1}\}).$$

Proof.

By the previous result,

$\text{Span}(\{v_1, \dots, v_n\}) \subset \text{Span}(\{v_1, \dots, v_{n+1}\})$. Now suppose that $x \in \text{Span}(\{v_1, \dots, v_{n+1}\})$. Then there are scalars a_k such that $x = a_1 v_1 + \dots + a_n v_n + a_{n+1} v_{n+1}$. But assumption there are also scalars b_k such that $v_{n+1} = b_1 v_1 + \dots + b_n v_n$. Therefore, $x = (a_1 + a_{n+1} b_1) v_1 + \dots + (a_n + a_{n+1} b_n) v_n \in \text{Span}(\{v_1, \dots, v_n\})$. This shows that $\text{Span}(\{v_1, \dots, v_{n+1}\}) \subset \text{Span}(\{v_1, \dots, v_n\})$ so the sets must be equal. \square

Enough

- 1 That is enough for today.